6.6 Mobius inversions

Let $X$ be a finite set.

Let $\mathcal{F} = \{ f : X \times X \to \mathbb{R} \mid \text{if } f(x, y) \neq 0, \text{ then } x \leq y \}$

Define the operation $*$ on $\mathcal{F}$ by

$$f * g = \begin{cases} \sum \{ f(x, z)g(z, y) \mid x \leq z \leq y \} & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Note $*$ is associative: $f * (g * h) = (f * g) * h$

Let $\delta = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$

Note $\delta$ acts as the identity for $*$: $f * \delta = \delta * f = f$

If $f(x, x) \neq 0$ for all $x \in X$, then $f$ is invertible: There exist $f^{-1}$ such that $f * f^{-1} = f^{-1} * f = \delta$. In this case,

$$f^{-1}(x, x) = \frac{1}{f(x, x)}$$

$$f^{-1}(x, y) = -\sum \{ f^{-1}(x, z) \frac{f(z, y)}{f(y, y)} \mid x \leq z < y \} \text{ for } x < y,$$

$$f^{-1}(x, y) = 0 \text{ if } x \not\leq y.$$