HW 5: Ch 4: 37, 44, 46, 48, 49, 51 (draw diagram for X_3 instead of H_4)

Defn: A *partial order* ($\leq$)is reflexive, anti-symmetric, and transitive.

Defn: A *strict partial order* ($<$)is irreflexive, anti-symmetric, and transitive.

Note: If $\leq \ \subset \ X \times X$ is a partial order, then
$< \ = \ \leq$ - the diagonal is a strict partial order.

Defn: An *equivalence relation* is reflexive, symmetric, and transitive.

Defn: x and y are *comparable* if xRy or yRx. Else x and y are *incomparable.*

Defn: A *total order* is a partial order where every pair of elements of X are comparable.

Thm 4.5.1: Suppose $|X| = n$. Then there exists a bijection between the total orders of X and the permutations of X. Hence there exists $n!$ different total orders on n.

Proof: Suppose $X = \{1, ..., n\}$ and suppose $f(1), f(2), ..., f(n)$ is a permutation of the elements of X.

Claim: $f(1) \leq f(2) \leq \ ... \ \leq f(n)$ defines a total order.

Note the above claim is equivalent to:

Claim: $f(i) \leq f(j)$ iff $i \leq j$ defines a total order on X.

Proof of claim:

Claim: $\leq$ is reflexive. That is, $\forall x \in X$, $x \leq x$.

Claim: $\leq$ is anti-symmetric. That is, if $x \leq y$ and $y \leq x$, then $x = y$.

Claim: $\leq$ is transitive. That is, if $x \leq y$ and $y \leq z$, then $x \leq z$.

Thus $\leq$ is a partial order.

Note every pair of elements of X is comparable. Thus $\leq$ is a total order.

Suppose we have a total order $\leq$ on X.

Claim: We can order the elements of X as follows:

$$f(1) \leq f(2) \leq ... \leq f(n) \text{ for some permutation of } X.$$

Proof by induction on $n = |X|$.

Suppose $n = 1$:

Suppose that if $|X| = n-1$, we can order the elements of X as follows: $f(1) \leq f(2) \leq ... \leq f(n-1)$ for some permutation of X.

Suppose $|X| = n$.

Note that we have shown a 1:1 correspondence between permutations of X and total orders of X. Hence there exists $n!$ different total orders on n.