New Office Hours: M 11:45am - 1:15pm, T 4:45pm - 5+,
WF 9:40 - 10:10am, Th 2:30 - 3:15pm, and by appointment.

Thm 2.1.1. Pigeonhole Principle (weak form): If you have $n+1$ pigeons in n pigeonholes, then at least one pigeonhole with be occupied by 2 or more pigeons.

Thm 2.1.1 If $f : A \to B$ is a function and $|A| = n+1$, and $|B| = n$, then f is not 1:1.

Cor: If $f : A \to B$ is a function and A is finite and $|A| > |B|$, then f is not 1:1.

Note that the domain must have more elements then the codomain to **guarantee** that f is not 1:1 as the following example illustrates: $\quad id : \{1, ..., n\} \to \{1, ..., n\},\ id(k) = k$ is $1:1$.

Recall that the *converse* of [p implies q] is [q implies p].
Note the converse of a theorem is frequently false as the following example illustrates:

$$c : \{1, ..., n\} \to \{1, ..., n\},\ id(k) = 1 \text{ is not } 1:1,$$

but domain does not have more elements than the codomain.

$f : A \to B$ a function which is not 1:1 does not imply $|A| > |B|$.

The *contrapositive* of [p implies q] is [not q implies not p].
The contrapositive of a theorem is true:

Cor: If $f : A \to B$ is a function which is 1:1, then $|A| \leq |B|$.

Related theorem:
Thm AB: Thm 2.1.1 If $f : A \to B$ is a function and
if $|A| = n = |B|$, then f is 1:1 iff f is onto.

Application 6: Chinese remainder theorem:
Suppose $m, n, a, b \in \mathcal{Z}$, $(m, n) = 1$, $0 \leq a \leq m-1$, $0 \leq b \leq n-1$, then $\exists x \geq 0$ such that $x = pm + a = qn + b$ for $p, q \in \mathcal{Z}$.

Moreover can take $p \in \{0, ..., n-1\}$.

Scratch work:

a is the remainder when x is divided by m.
b is the remainder when x is divided by n.

$x = a \bmod m, \quad x = b \bmod n.$

Proof plus thoughts:

We need to use the Pigeonhole principle (or related theorem). Thus we need to create objects. We are interested in $pm+a$ for some unknown $p \in \mathcal{Z}$. Thus one idea is to create the following objects:

$\mathcal{O} = \{a, m+a, 2m+a, ..., (n-1)m+a\}$.

Note $\mathcal{O}$ has ________ distinct objects.

We need to create boxes. What else are we interested in? How about remainders?

Let r_k = the remainder of $km + a$ when divided by n.

Properties of r_k:

Thm 2.2.1 Pigeonhole Principle (strong form): Let $q_1, q_2, ..., q_n$ be positive integers. If $q_1 + q_2 + ... + q_n - n + 1$ objects are put into n boxes, then for some i the ith box contains at least q_i objects

Proof Outline:

Cor: Pigeonhole Principle (weak form):

Proof. Let $q_i = 2$ for all i.

Cor: If $n(r-1)+1$ objects are put into n boxes, then there exists a box containing at least r objects.

Proof: Let $q_i = r$ for all i. Note $nr - n + 1 = n(r-1)+1$.

Cor A: If $m_i \in \mathcal{Z}_+$ and if $\frac{m_1+...+m_n}{n} > r-1$, then there exists an i such that $m_i \geq r$.

Cor A: If $m_i \in \mathcal{Z}_+$ and if $\frac{m_1+...+m_n}{n} \geq r$, then there exists an i such that $m_i \geq r$.

Lemma B: If $\frac{m_1+...+m_n}{n} < r$, then there exists an i s. t. $m_i < r$.

Appl: Suppose you have 20 pairs of shoes in your closet. If you grab n shoes at random, what should n be so that you are guaranteed to have a matching pair of shoes.

Appl: Suppose you have 20 pairs of socks. If you grab n socks at random, what should n be so that you are guaranteed to have a matching pair of shoes.

Appl: Suppose you have 20 pairs of socks. If 7 are black and 13 are white, and if you grab n socks at random, what should n be so that you are guaranteed to have a pair of socks of the same color.

Appl 7: If you have an arbitrary number of apples, bananas and oranges, what is the smallest number of these fruits that one needs to put in a basket in order to guarantee there are at least 8 apples or at least 6 bananas or at least 9 oranges in the basket.