2.5 Combinations of Multisets

Thm 2.5.1 Let $S = \{\infty \cdot a_1, ..., \infty \cdot a_k\}$. Then the number of $r$-combinations of $S$ is

Proof: The number of $r$-combinations of $S$

= the number of integral solutions to the equation

$x_1 + x_2 + \ldots + x_k = r$  \( (*) \)

where $x_i \geq 0 \ \forall i$ (and where $x_i =$ the number of $a_i$’s chosen for an $r$–combination).

= the number of permutations of $\{r \cdot 1, (k - 1) \cdot +\}$ by the following:

Suppose $(c_1, c_2, ..., c_k)$ is a solution to \( (*) \). This corresponds to the permutation $11...1 + 1.1 + \ldots + 11...1$,

where there are $k - 1$ ’s and $c_1$ 1’s between the $(i-1)$th and $i$th +’s for $i = 2, ..., k - 1$, and $c_k$ 1’s after the last +. Since $c_1 + c_2 + \ldots + c_k = r$, there are $r$ 1’s, and thus $11...1 + 1.1 + \ldots + 11...1$ is a permutations of $\{r \cdot 1, (k - 1) \cdot +\}$.

A permutation of $\{r \cdot 1, (k - 1) \cdot +\}$ corresponds to a solution $(c_1, c_2, ..., c_k)$ of \( (*) \) where $c_1 =$ the number of 1’s before the first +, $c_i =$ the number of 1’s between the $(i-1)$th and $i$th +’s for $i = 2, ..., k - 1$, and $c_k =$ the number of 1’s after the last +. Since there are $r$ 1’s, $c_1 + c_2 + \ldots + c_k = r$.

The number of permutations of $\{r \cdot 1, (k - 1) \cdot +\}$ is

Corollary: Let $S = \{r \cdot a_1, ..., r \cdot a_k\}$. Then the number of $r$-combinations of $S$ is

Proof:

Some examples

$S = \{\infty \cdot a_1, \infty \cdot a_2, ..., \infty \cdot a_5\}$.

Then a 4-combination of $S$ is $\{a_3, a_3, a_3, a_5\}$

Suppose $x_1 + x_2 + x_3 + x_4 + x_5 = 4$.
Then $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 3, 0, 1)$ is a solution.

$11 + 111 + 1$ is a permutation of $\{4 \cdot 1, (5 - 1) \cdot +\}$

$(x_1, x_2, x_3, x_4, x_5) = (2, 1, 0, 1, 0)$ is a solution to
$x_1 + x_2 + x_3 + x_4 + x_5 = 4$.

$11 + 1 + +11$ is a permutation of $\{4 \cdot 1, (5 - 1) \cdot +\}$

A 4-combination of $S$ is $\{a_1, a_1, a_2, a_4\}$

$11 + 1 + +1$ is a permutation of $\{4 \cdot 1, (5 - 1) \cdot +\}$

A 4-combination of $S$ is $\{a_5, a_5, a_5, a_5\}$

$(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 0, 4)$ is a solution to
$x_1 + x_2 + x_3 + x_4 + x_5 = 4$.