14.2: Burnside's Theorem.

Defn: $\mathcal{C}(f) = \{\mathbf{c} \in \mathcal{C} \mid f * \mathbf{c} = \mathbf{c}\}$.

Thm 14.2.3:
Suppose for all $f \in G$ and for all $\mathbf{c} \in \mathcal{C}$, $f * \mathbf{c} \in \mathcal{C}$. Then

$N(G, C) =$ the number of non-equivalent colorings in $\mathcal{C}$

$$= \frac{1}{|G|} \sum_{f \in G} |\mathcal{C}(f)|$$

$=$ the average # of colorings fixed by a permutations in G.

We will calculate the number of circular permutations as well as the number of bracelets containing 4 objects using 2 colors.

Note that in most cases, the number of circular permutations does NOT equal the number of bracelets.

Find the number of circular permutations of 4 elements from $\{4 \cdot blue, 4 \cdot red\}$.

$$X = \{1, 2, 3, 4\}, \qquad \mathcal{C} = \{f : X \to \{blue, red\}\}, \qquad |\mathcal{C}| = 2^4.$$

For circular permutations of a regular n-gon,

$$G = \text{group of rotations.}$$

$$G = \{id, \rho_4, \rho_4^2, \rho_4^3\} \qquad |G| = 4.$$

Need to calculate $|\mathcal{C}(f)|$ for each $f \in G = \{id, \rho_4, \rho_4^2, \rho_4^3\}$

$|\mathcal{C}(id)| = |\mathcal{C}| = 2^4 = 16$

$|\mathcal{C}(\rho_4)| = 2$ since ρ only preserves a coloring when all vertices have the same color (i.e., all blue or all red).

$|\mathcal{C}(\rho_4^2)| = 2^2 = 4$ since ρ^2 only preserves a coloring when vertices on diagonal corners have the same color. Thus if we choose colors for vertices 1 and 2, then the colors of vertices 3 and 4 are determined.

$|\mathcal{C}(\rho_4^3)| = 2$ (similar to ρ_4 case).

Thus the number of circular permutations of 4 elements from $\{4 \cdot blue, 4 \cdot red\} =$

$$N(G, C) = \frac{1}{|G|} \sum_{f \in G} |\mathcal{C}(f)| = \frac{1}{4}(16 + 2 + 4 + 2) = \frac{1}{4}(24) = 6$$

Find the number of bracelets one can make using 4 beads if you have red and blue beads.

$$X = \{1, 2, 3, 4\}, \qquad \mathcal{C} = \{f : X \to \{blue, red\}\}, \qquad |\mathcal{C}| = 2^4.$$

For bracelets, G = group of rotations and reflections.

$$G = \{id, \rho_4, \rho_4^2, \rho_4^3, \tau_1, \tau_2, \tau_3, \tau_4\} \qquad |G| = 8.$$

Calculate $|\mathcal{C}(f)|$ for each $f \in G = \{id, \rho_4, \rho_4^2, \rho_4^3, \tau_1, \tau_2, \tau_3, \tau_4\}$

From previous example,

$$|\mathcal{C}(id)| = 16 \qquad |\mathcal{C}(\rho_4)| = 2 \qquad |\mathcal{C}(\rho_4^2)| = 4 \qquad |\mathcal{C}(\rho_4^3)| = 2$$

$|\mathcal{C}(\tau_1)| = 2^3 = 8$. Since τ_1 reflects across the diagonal joining vertices 1 and 3, the color of vertex 2 must be the same as the color of vertex 4. Thus there are 2 choices of colors for each of the vertices 1, 2, 3, but only 1 choice for vertex 4.

$|\mathcal{C}(\tau_2)| = 2^3 = 8$ (Similar to τ_1)

$|\mathcal{C}(\tau_3)| = 2^2 = 4$. A reflection across a vertical line means (vertex 2 must have the same color as vertex 1) and (vertex 4 must have the same color as vertex 3).

$|\mathcal{C}(\tau_4)| = 2^2 = 4$ (Similar to τ_3)

Thus the number of circular permutations of 4 elements from $\{4 \cdot blue, 4 \cdot red\}$ =

$$N(G, C) = \frac{1}{|G|} \sum_{f \in G} |\mathcal{C}(f)| = \frac{1}{8}(16+2+4+2+8+8+4+4) = 6$$