\[ C^\infty(M) = \{ g \mid g^{smooth} : M \to \mathbb{R} \} \]

\[ D \text{ is a derivation iff } D : C^\infty(p) \to \mathbb{R} \text{ and } D \text{ is linear and satisfies the Leibniz rule.} \]

That is \( D \) is a derivation if 
\[ D(f) \in \mathbb{R}, \quad D(cf) = cD(f), \quad D(f + g) = D(f) + D(g), \]
\[ D(fg) = f(p)Dg + g(p)Df \]

Defn: A vector field or section of the tangent bundle \( TM \) is a smooth function
\( s : M \to TM \) so that \( \pi \circ s = id \) [i.e., \( s(p) = (p, v_p) \)].

Ex: If \( M = \mathbb{R} \), let \( s(p) = (p, (\frac{d}{dx})_p) \)

Sometimes we will drop the \( p \) and write \( s(p) = (\frac{d}{dx})_p \)

Let \( f \in C^\infty(\mathbb{R}) \). For all \( p \in \mathbb{R} \), 
\[ s(p)(f) = (\frac{df}{dx})_p = \frac{df}{dx}(p) \]

Define \( s_f : \mathbb{R} \to \mathbb{R} \), \( s_f(p) = \frac{df}{dx}(p) \). I.e., \( s_f = \frac{df}{dx} \)

Note \( s_f \) is smooth.

Lemma 3.4.1: For any vector field \( s \) and smooth functions \( f \) and \( g \) on \( M \), we have 
\[ s_{fg}(p) = f(p) \cdot s_g(p) + s_f(p) \cdot g(p) \]

Proof: 
\[ \frac{d(fg)}{dx}(p) = f(p)\frac{dg}{dx}(p) + \frac{df}{dx}(p)g(p) \]

We can think of a vector field as a function 
\( S : C^\infty(M) \to C^\infty(M), \ S(f) = s_f \)

Ex: \( S : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \ S(f) = \frac{df}{dx} \). I.e., \( S = \frac{d}{dx} \)
Ex: If $M = \mathbb{R}$, then $s(p) = a(p)\left(\frac{d}{dx}\right)_{p}$ where $a : \mathbb{R} \to \mathbb{R}$ is a smooth function.

Let $f \in C^\infty(\mathbb{R})$. For all $p \in \mathbb{R}$, $s(p)(f) = a(p)\left(\frac{df}{dx}\right)_{p} = a(p)\frac{df}{dx}(p)$

Define $s_f : \mathbb{R} \to \mathbb{R}$, $s_f(p) = a(p)\frac{df}{dx}(p)$. I.e., $s_f = a\frac{df}{dx}$

Note $s_f$ is smooth.

Lemma 3.4.1: For any vector field $s$ and smooth functions $f$ and $g$ on $M$, we have

$s_{fg}(p) = f(p) \cdot s_g(p) + s_f(p) \cdot g(p)$

Proof: $a(p)\frac{d(fg)}{dx}(p) = a(p)f(p)\frac{dg}{dx}(p) + a(p)\frac{df}{dx}(p)g(p)$

We can think of a vector field as a function

$S : C^\infty(M) \to C^\infty(M), S(f) = s_f$

Ex: $S : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), S(f) = a\frac{df}{dx}$ I.e., $S = a\frac{d}{dx}$
In the above we used the charts \( \phi_p : \mathbb{R} \rightarrow \mathbb{R}, \phi_p(x) = x - p \).

Thus \( \frac{d(g(\phi_p^{-1}(x)))}{dx}\big|_{x=0} = \frac{d(g(x+p))}{dx}\big|_{x=0} = \frac{dg}{dx}(p) \)

Note \( \phi_0(x) = \phi_p(x + p) \).

Thus \( \frac{d(\phi_p(\phi_0^{-1}(x)))}{dx}\big|_{x=0} = \frac{d(\phi_p(\phi_p^{-1}(x+p)))}{dx}\big|_{x=0} = \frac{d(x+p)}{dx}\big|_{x=0} = 1 \)

If we use the chart \( \psi_q : \mathbb{R} \rightarrow \mathbb{R}, \psi_q(x) = q - x \).

Then \( \frac{d(g(\psi_q^{-1}(x)))}{dx}\big|_{x=0} = \frac{d(g(p-x))}{dx}\big|_{x=0} = -\frac{dg}{dx}(p) \)

Note \( \frac{d(\psi_q(x+p))}{dx}\big|_{x=0} = \frac{d\psi_q}{dx}\big|_{p} = \frac{d(q-x)}{dx}\big|_{p} = -1 \)

Example of a non-smooth vector field on \( \mathbb{R} \):

If \( p \geq 0 \), let \( s(p) = (p, (\frac{d}{dx})_p) \)
[i.e., the basis element of \( T_p(\mathbb{R}) \) from \( \phi_p \)]

If \( p < 0 \), let \( s(p) = (p, (\frac{d}{dx})_p) \)
[i.e., the basis element of \( T_p(\mathbb{R}) \) from \( \psi_p \)]
Ex: If $M = \mathbb{R}^2$, then $s(p) = a(p)\left(\frac{\partial}{\partial x}\right)_p + b(p)\left(\frac{\partial}{\partial y}\right)_p$ where $a, b : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions.

Ex: Let $\{(\frac{\partial}{\partial x_1})_p, \ldots, (\frac{\partial}{\partial x_m})_p\}$ be a basis for $T_p(M)$.

Let $s : M \to TM$, $s(p) = (p, \sum_{i=1}^{m} a_i(p)(\frac{\partial}{\partial x_i})_p)$