$TM = \cup_{p \in M} T_p(M) = \{(p, v) \mid p \in M, v \in T_pM\}$,

let $\pi\colon TM \to M$ be defined by $\pi(p, v) = p$.

Let (ϕ, U) be a chart for M.

If $q \in U$, let $\{(\frac{\partial}{\partial x_1})_q, ..., (\frac{\partial}{\partial x_m})_q\}$ be a basis (w.r.t (ϕ, U)) for $T_q(M) = T_q$

$t_\phi\colon \pi^{-1}(U) \to \phi(U) \times \mathbf{R}^m \subset \mathbf{R}^{2m}$,

$t_\phi(q, v) = (\phi(q), a_1, ..., a_m)$ where $v = \Sigma_{i=1}^m a_i (\frac{\partial}{\partial x_i})_q$

Let $\mathcal{A}$ be a maximal atlas for M.

Basis for topology on TM :
$\{W \mid \exists (\phi, U) \in \mathcal{A}$ s.t. $W \subset \pi^{-1}(U)$ and $t_\phi(W)$ open in $\mathbf{R}^{2m}\}$

Claim: TM is a $2m-$manifold and
$\mathcal{C} = \{(t_\phi, \pi^{-1}(U)) \mid (\phi, U) \in \mathcal{A}\}$ is a pre-atlas for TM.

$\pi\colon TM \to M$, $\pi(p, v) = p$ is smooth

$df : TM \to TN$ defined by $df(p, v) = (f(p), d_pf(v))$ is smooth if $f : M \to N$ is smooth.

Proof: See Hitchin 4.1 (in Chapter 1 of
http://www2.maths.ox.ac.uk/ hitchin/hitchinnotes/hitchinnotes.

Defn: A *vector field* or *section of the tangent bundle TM* is a smooth function
s: $M \to TM$ so that $\pi \circ s = id$ [i.e., $s(p) = (p, v_p)$].

Defn: s is *never zero* if $s(p) \neq (p, \mathbf{0})$ for all $p \in M$.

Prop: Let G be a Lie group. Then G admits a never-zero vector field.

Note: S^n admits a never-zero vector field iff n odd.

Let $p_2(s(p)) = p_2(p, v_p) = v_p$

Defn: The vector fields $s_1, ..., s_k$ are *linearly independent* iff for all $p \in M$, $p_2(s_1(p)), ..., p_2(s_k(p))$ are linearly independent.

Prop: M is parallelizable (or equivalently the "tangent bundle π: $TM \to M$ is trivial") iff TM admits m linearly independent vector fields.

Defn: A *flow* on M is a smooth action of the Lie group $\mathbf{R}^1$ on M, σ: $\mathbf{R}^1 \times M \to M$.

A flow is also called a *dynamical system.*

A *flow line* is the smooth path $\alpha_p : \mathbf{R} \to M$, $\alpha_p(t) = \sigma(t, p)$. ∎