$TM = \bigcup_{p \in M} T_p(M) = \{(p, v) \mid p \in M, v \in T_pM\}$,

let $\pi: TM \to M$ be defined by $\pi(p, v) = p$.

Let $(\phi, U)$ be a chart for $M$.

If $q \in U$, let \{($\frac{\partial}{\partial x_1})_q, ..., (\frac{\partial}{\partial x_m})_q$\} be a basis (w.r.t $(\phi, U)$) for $T_q(M) = T_q$

$t_\phi: \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^m \subset \mathbb{R}^{2m},$

$t_\phi(q, v) = (\phi(q), a_1, ..., a_m)$ where $v = \Sigma_{i=1}^m a_i (\frac{\partial}{\partial x_i})_q$

Let $\mathcal{A}$ be a maximal atlas for $M$.

Basis for topology on $TM:$
\{\{W \mid \exists (\phi, U) \in \mathcal{A} \text{ s.t. } W \subset \pi^{-1}(U) \text{ and } t_\phi(W) \text{ open in } \mathbb{R}^{2m}\}\}

Claim: $TM$ is a $2m$–manifold and
$\mathcal{C} = \{(t_\phi, \pi^{-1}(U)) \mid (\phi, U) \in \mathcal{A}\}$ is a pre-atlas for $TM$.

$\pi: TM \to M, \pi(p, v) = p$ is smooth

$df : TM \to TN$ defined by $df(p, v) = (f(p), d_pf(v))$ is smooth if $f : M \to N$ is smooth.

Proof: See Hitchin 4.1 (in Chapter 1 of
http://www2.maths.ox.ac.uk/~hitchin/hitchinnotes/hitchinnotes.)
Defn: A vector field or section of the tangent bundle $TM$ is a smooth function 
$s: M \to TM$ so that $\pi \circ s = id$ [i.e., $s(p) = (p, v_p)$].

Defn: $s$ is never zero if $s(p) \neq (p, 0)$ for all $p \in M$.

Prop: Let $G$ be a Lie group. Then $G$ admits a never-zero vector field.

Note: $S^n$ admits a never-zero vector field iff $n$ odd.

Let $p_2(s(p)) = p_2(p, v_p) = v_p$

Defn: The vector fields $s_1, \ldots, s_k$ are linearly independent iff for all $p \in M$, $p_2(s_1(p)), \ldots, p_2(s_k(p))$ are linearly independent.

Prop: $M$ is parallelizable (or equivalently the “tangent bundle $\pi: TM \to M$ is trivial”) iff $TM$ admits $m$ linearly independent vector fields.

Defn: A flow on $M$ is a smooth action of the Lie group $\mathbb{R}^1$ on $M$, $\sigma: \mathbb{R}^1 \times M \to M$.

A flow is also called a dynamical system.

A flow line is the smooth path $\alpha_p: \mathbb{R} \to M$, $\alpha_p(t) = \sigma(t, p)$. 