Thm 3.3: Suppose F is a continuous vector field.

$F = \nabla f$ iff F has path independent line integrals.

Moreover if C is a piecewise C^1 curve, then

$$\int_C F \cdot ds = f(B) - f(A)$$

where A is the initial point of C and B is the terminal point of C.

Thm 3.5. Suppose F is a C^1 vector field and suppose $R =$ the domain of F is simply connected in $\mathbf{R}^2$ or $\mathbf{R}^3$.Then

$F = \nabla f$ for $f \in C^2$ iff $\nabla \times F = 0$ for all $x \in R$.

Suppose $F = (M(x,y), N(x,y))$. Then $\nabla \times F = (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mathbf{k}$

Ex: $F(x,y) = (x^3, e^y)$

Parametrized curves:

Ex: $f : [0, 2\pi] \to \mathbf{R}^2$, $f(t) = (cos(t), sin(t))$

Note this is a function of 1 variable. Thus 1 degree of freedom. Hence we obtain 1-dimensional curves.

Note f is 1:1 on $(0, 2\pi)$ (but not 1:1 on boundary of $[0, 2\pi]$

Thus the image of $f = \{(cos(t), sin(t)) \mid t \in \mathbf{R}\}$ is a curve in $\mathbf{R}^2$.

A parametrization of the image of f is

$$x(t) = cos(t), \quad y(t) = sin(t).$$

This curve can also be represented by the level set, $g^{-1}(1)$ where $g(x, y) = x^2 + y^2$

The graph of $f = \{(t, f(t)) = (t, cos(t), sin(t)) \mid t \in \mathbf{R}\}$ is also a curve in $\mathbf{R}^3$.

A parametrization of the graph of f is

$$x(t) = t, \quad y(t) = cos(t), \quad z(t) = sin(t).$$

7.1 Parametrized surfaces

Ex: $f(s, t) = [0, 2\pi] \times \mathbf{R} \to \mathbf{R}^3$

$f(s, t) = (cos(s), sin(s), t)$

Note this is a function of 2 variables. Thus 2 degrees of freedom. Hence the image is a 2-dimensional surface.

Note f is 1:1 on the interior of the domain, but not on the boundary.

The graph of f is also a 2-dimensional surface (in $\mathbf{R}^5$), but we will focus on the image of f.

Defn: Suppose $X : D \to \mathbf{R}^n$, $D \subset \mathbf{R}^2$.

Fix $t_0 \in \mathbf{R}$. The *s-coordinate curve at* $t = t_0$ is the image of the map $c_1(s) = X(s, t_0)$.

Fix $s_0 \in \mathbf{R}$. The *t-coordinate curve at* $s = s_0$ is the image of the map $c_2(t) = X(s_0, t)$.

Suppose $X(s,t)$ differentiable.

Let $T_s(s_0,t_0) = \frac{\partial X}{\partial s}(s_0,t_0) =$ tangent vector to the s-coordinate curve $X(s,t_0)$

Let $T_t(s_0,t_0) = \frac{\partial X}{\partial t}(s_0,t_0) =$ tangent vector to the t-coordinate curve $X(s_0,t)$

Thus T_s and T_t are tangent to the surface $X(D)$

A normal to this surface is

Defn: A parametrized surface $S = X(D)$ is *smooth* at $X(s_0,t_0)$ if X is C^1 near (s_0,t_0) and if $N(s_0,t_0) = T_s(s_0,t_0) \times T_t(s_0,t_0) \neq 0$.

If S is smooth at every point in D, then the surface S is *smooth.*

If S is a smooth parametrized surface, then $N = T_s \times T_t$ is the *standard normal vector arising from the parametrization of* X.

Let V be a finite-dimensional vector space over $\mathbf{R}$.

The dual of $V = V^* = \{f : V \to \mathbf{R} \mid f \text{ linear }\}$

Note V^* is a vector space. The elements of V^* are called *covectors.*

If $e_1, ..., e_n$ basis for V, then $w_1, ..., w_n$ basis for V^* where $w_i : V \to \mathbf{R}$ where $w_i(e_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

dim V = dim V^*

Let $F_* : V \to W$ be a linear map between vector spaces
The dual map map is $F^* : W^* \to V^*$, $F(g) = g \circ F$.

F_* is injective implies F^* injective

F_* is surjective implies F^* surjective

$(G_* \circ F_*)^* = F^* \circ G^*$.

$d : V \to (V^*)^*$, $d(v) = h$ where $h : V^* \to R$, $h(f) = f(v)$.

Thus $(V^*)^*$ is naturally isomorphic to V.

Defn: The dual of $T_pM = T_p^*M$ is the *cotangent space* to M at p.

If $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}$ is a basis for T_pM, then the dual basis will be denoted $dx_1, ..., dx_m$.

B is bilinear if
$B(cv_1 + dv_2, w) = cB(v_1, w) + dB(v_2, w)$
$B(v, cw_1 + dw_2) = cB(v, w_1) + dB(v, w_2)$

Thus

$B((v_1, w_1) + (v_2, w_2)) = B(v_1 + v_2, w_1 + w_2)$
$= B(v_1, w_1 + w_2) + B(v_2, w_1 + w_2)$
$= B(v_1, w_1) + B(v_1, w_2) + B(v_2, w_1) + B(v_2, w_2)$

B is linear if
$B((v_1, w_1) + (v_2, w_2)) = B((v_1, w_1)) + B((v_2, w_2))$
$B(c(v_1, w_1)) = cB((v_1, w_1)$