Randell 2.1

Let $p \in M$.

Suppose $g_i : U_i \longrightarrow N$, where $p \subset U_i^{open} \subset M$.

$g_1 \sim g_2$ if $\exists V$ such that $p \in V \subset U_1 \cap U_2$ and $g_1(x) = g_2(x)$ $\forall x \in V$.

The equivalence class [g] is a *germ*.

$G(p, N) = \{[g] \mid g^{smooth} : U \longrightarrow N$, for some $U^{open}$ such that $p \in U \subset M\}$

$G(p) = G(p, \mathbf{R})$

$G(p)$ is an algebra over $\mathbf{R}$.

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Let $\alpha : I \longrightarrow M$ where $I =$ an interval $\subset \mathbf{R}$, $\alpha(0) = p$.

Note $[\alpha] \in G[0, M]$

Directional derivative of $[g]$ in direction $[\alpha] =$

$$D_\alpha g = \frac{d(g \circ \alpha)}{dt}|_{t=0} \in \mathbf{R}$$

Note $D_\alpha : G(p) \longrightarrow \mathbf{R}$ is linear and satisfies the Leibniz rule.

Boothby 2.4

$T_{\mathbf{a}}(\mathbf{R}^n) = \{(\mathbf{a}, \mathbf{x}) \mid \mathbf{x} \in \mathbf{R}^n\}$, $\phi(\mathbf{ax}) = \mathbf{x} - \mathbf{a}$

canonical basis $\{E_{i\mathbf{a}} = \phi^{-1}(e_i) \mid i = 1, ..., n\}$

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Let $\mathbf{a} \in R^n$

Suppose $f : X \subset \mathbf{R}^n \to \mathbf{R}$ where $\mathbf{a} \subset X^{open} \subset M$.

$f \sim g$ if $\exists U^{open}$ s.t. $\mathbf{a} \in U$ and $f(x) = g(x) \forall x \in U$.

Let $C^\infty(a) = \{f : X \subset \mathbf{R}^n \to \mathbf{R} \in C^\infty \mid a \in dom f\}$

$f_i : U_i \to \mathbf{R} \in C^\infty(a)$ implies $f_1 + f_2 : U_1 \cap U_2 \to \mathbf{R} \in C^\infty(a)$, $\alpha f_i : U_i \to \mathbf{R} \in C^\infty(a)$, and $f_1 f_2 : U_1 \cap U_2 \to \mathbf{R} \in C^\infty(a)$,

Thus $C^\infty(a)$ is an algebra over $\mathbf{R}$

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Let $X_{\mathbf{a}} = \Sigma_{i=1}^n \xi_i E_{i\mathbf{a}}$

$X^*_{\mathbf{a}} : C^\infty(\mathbf{a}) \to \mathbf{R}$

$X^*_{\mathbf{a}}(f) = \Sigma_{i=1}^n \xi_i \frac{\partial f}{\partial x_i}|_{\mathbf{a}}$ = directional derivative of $f$ at $\mathbf{a}$ in the direction of $X_{\mathbf{a}}$.

Let $x_j : \mathbf{R}^n \to \mathbf{R}$, $x_j(\mathbf{x}) = x_j \quad X^*_{\mathbf{a}}(x_j) = \Sigma_{i=1}^n \xi_i \frac{\partial x_j}{\partial x_i}|_{\mathbf{a}} = \xi_i$

$X^*_{\mathbf{a}}$ is linear and satisfies the Leibniz rule.

Let $T_p(M) = \{v : G(p) \to \mathbf{R} \mid v$ is linear and satisfies the Leibniz rule $\}$

If $v \in T_p(M)$ is called a *derivation*

$D_\alpha \in T_p(M)$

$T_p(M)$ is closed under addition and scalar multiplication and hence is a vector space over $\mathbf{R}$

Find a basis for $T_p(M)$:

Let $(U, \phi)$ be a chart at $p$ such that $\phi : U^{open} \to \phi(U)^{open} \subset R^m$ is a homeomorphism, $\phi(p) = \mathbf{0} \in R^m$

$\mathbf{0} \in \phi(U)^{open}$ implies $\exists \epsilon > 0$ such that $B_\epsilon(\mathbf{0}) \subset \phi(U)$

Thus if $t \in (-\epsilon, \epsilon)$, then $(0, ..., t, ..., 0) \subset \phi(U)$.

Define $\alpha_i : (-\epsilon, \epsilon) \to M$, $\alpha_i(t) = \phi^{-1}(0, ..., t, ..., 0)$

Let $v_i = D_{\alpha_i}$.

Claim: $\{v_1, ..., v_m\}$ is a basis for $T_p(M)$.

Let $\mathcal{D}(a) = \{D : C^\infty(\mathbf{a}) \to \mathbf{R} \mid D$ is linear and satisfies the Leibniz rule $\}$

$D \in \mathcal{D}(a)$ is called a *derivation*

$X^*_{\mathbf{a}} \in \mathcal{D}(a)$

$\mathcal{D}(a)$ is closed under addition and scalar multiplication and hence is a vector space over $\mathbf{R}$

Let $j : T_{\mathbf{a}}(\mathbf{R}^n) \to \mathcal{D}(a)$, $j(X_{\mathbf{a}}) = X^*_{\mathbf{a}}$

Claim: $j$ is an isomorphism.

Let $X_{\mathbf{a}} = \Sigma_{i=1}^n \xi_i E_{i\mathbf{a}}$ and $Z_{\mathbf{a}} = \Sigma_{i=1}^n \zeta_i E_{i\mathbf{a}}$

$j$ is a homomorphism.

$j$ is 1-1:

If $j(X_{\mathbf{a}}) = j(Z_{\mathbf{a}})$, then $X^*_{\mathbf{a}}(x_j) = \Sigma_{i=1}^n \xi_i \frac{\partial x_j}{\partial x_i}|_{\mathbf{a}} = \xi_i = \zeta_i = Z^*_{\mathbf{a}}(x_j)$

$j$ is onto: Let $D$ be a derivation.

Suppose $f(\mathbf{x}) = 1$. Then $Df = 0$ by product rule.
Suppose $g(\mathbf{x}) = c$. Then $Dg = D(cf) = cDf = 0$

Let $h_i(\mathbf{x}) = x_i$. Let $\xi_i = Dh_i$. Then $D = X^*_{\mathbf{a}}$ where $X_{\mathbf{a}} = \Sigma_{i=1}^n \xi_i E_{i\mathbf{a}}$ (proof: long calculation, see Boothby).

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Note since $X^*_{\mathbf{a}}(f) = \Sigma_{i=1}^n \xi_i \frac{\partial f}{\partial x_i}|_{\mathbf{a}}$, $j(E_{i\mathbf{a}}) = E_{i\mathbf{a}}* = \frac{\partial}{\partial x_i}|_{\mathbf{a}}$,