Thm. $f : M \to N$ embedding implies $f(M)$ is a submanifold of $N$.

Recall $K$ is a submanifold of $N$ if $\forall q \in K \subset N$, $\exists g^{\text{smooth}} : V^{\text{open}} \subset N \to \mathbb{R}^{n-m}$, $q \in V$ such that $K \cap V = g^{-1}(0)$ and rank $d_p g = n - m$.

Proof. Since $f : M \to N$ embedding, $f : M \to N$ is a 1-1 immersion and

$f : M \to f(M)$ is a homeomorphism where $f(M)$ is a subspace of $N$

Take $q \in f(M)$.

Since $f$ is 1:1, $\exists! p \in M$ such that $f(p) = q$.

$f : M \to N$ an immersion implies $f$ has rank $m \leq n$.

Thus by the rank theorem,
Defn. Suppose \( f : M \to N \) is smooth.

\( p \in M \) is a critical point and \( f(p) \) is a critical value if \( \text{rank } d f_p < n \).

If \( p \in M \) is not a critical point, then it is a regular point.

If \( q \in N \) is not a critical value, then it is a regular value.

Note: \( q \in N \) is a regular value iff \( f^{-1}(q) = \emptyset \) or \( \forall p \in f^{-1}(q), df_p = n \).

Thm 2.3.13: Let \( q \) be a regular value of \( f : M \to N \). Then either \( f^{-1}(q) = \emptyset \) or \( f^{-1}(q) \) is an \((m-n)\)-submanifold of \( M \).

\( \text{Gl}(n, \mathbb{R}) \) is an \( n^2 \) manifold.

\( A \in \text{Gl}(n, \mathbb{R}) \) is orthogonal if \( A^t A = I \).

The orthogonal group = \( O(n) = \{ A \in \text{Gl}(n, \mathbb{R}) \mid A^t A = I \} \)

The special orthogonal group = \( SO(n) = \{ A \in O(n) \mid \text{det}(A) = 1 \} \)

\( O(n), SO(n) \) are subgroups of \( \text{Gl}(n, \mathbb{R}) \).

\( O(n), SO(n) \) are closed in \( \text{Gl}(n, \mathbb{R}) \).

If \( A \in O(n) \), then \( \text{det}(A) = \pm 1 \)

\( SO(n) \) is open in \( 0(n) \).

\( s : \text{Gl}(n, \mathbb{R}) \to \text{Gl}(n, \mathbb{R}), s(A) = A^t A \) is smooth.

Let \( S = \) the set of symmetric matrices.

Then \( S = \) is an \( n^2 \) manifold.

\( s : \text{Gl}(n, \mathbb{R}) \to S, s(A) = A^t A \) is smooth.

\( s^{-1}(I) = \)

Claim: \( I \) is a regular value of \( s : \text{Gl}(n, \mathbb{R}) \to S, s(A) = A^t A \).

That is, if \( A \in O(n) \), \( d_A S \) has rank \( \frac{n(n+1)}{2} \).

\( n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2} \).

Thus if \( I \) is a regular value, \( O(n) \) is an \( \frac{n(n-1)}{2} \) submanifold of \( \text{Gl}(n, \mathbb{R}) \).