$T_p(M) = \{v : G(p) \to \mathbf{R} \mid v$ is linear and satisfies the Leibniz rule $\}$

$G(p) = \{[g] \mid g^{smooth} : U \to \mathbf{R}$, for some U^{open} such that $p \in U \subset M\}$ is an algebra over $\mathbf{R}$.

$v \in T_p(M)$ is called a *derivation*

The directional derivative of $[g]$ in direction $[\alpha] =$

$$D_\alpha g = \frac{d(g \circ \alpha)}{dt}|_{t=0} = g'(\alpha(0))\alpha'(0) \in \mathbf{R}$$

Properties:

D_α is linear and satisfies the Leibniz rule. i.e,

1a.) $D_\alpha(g + h) = D_\alpha g + D_\alpha h$ 1b.) $D_\alpha(cg) = cD_\alpha g$

2.) $D_\alpha(g \cdot h) = D_\alpha g \cdot h(p) + g(p) \cdot D_\alpha h$

Thus $D_\alpha \in T_p(M)$

Thm: Let M be an m-manifold, then $T_p(M)$ is an m-dimensional real vector space. $[(c_1 v + c_2 w)(f) = c_1 v(f) + c_2 w(f)]$.

Take a chart (U, ϕ) at p where $\phi(p) = \mathbf{0}$,

The *standard basis* for $T_p(M)$ w.r.t.$(U, \phi) = \{v_1, ..., v_m\}$,

where $v_i = D_{\alpha_i}$ and

$\alpha_i : (-\epsilon, \epsilon) \to M$, $\alpha_i(t) = \phi^{-1}(0, ..., t, ..., 0)$ for some $\epsilon > 0$.

Prop: $\{v_1, ..., v_m\}$ are linearly independent.

Proof: Evaluate v_i at a “projection map”.

Thm: $\{v_1, ..., v_m\}$ span $T_p(M)$.

If $v \in T_p(M)$, then $v = \Sigma_{i=1}^m a_i v_i$ where $a_i = v([\pi_i \circ \phi])$

Prop: If v is a derivation, and f is constant, then $v(f) = 0$.

Suppose $f^{smooth} : M \to N$, $f(p) = q$.

The *tangent (or differential) map*,

$$df_p : T_p M \to T_q N$$

$df_p(v) =$ the derivation which takes $[g] \in G(q)$ to the real number $v([g \circ f])$

I.e., df_p takes the derivation $(v : G(p) \to \mathbf{R}) \in T_p M$

to the derivation in $T_q N$ which takes

the germ $[g] \in G(q)$ to the real number $v([g \circ f])$.

Suppose $f : M \to N$ is smooth. Then $f^* : G(q) \to G(p)$ is a homomorphism.

Let V be a real vector space.
Then the dual space, $V^* = \{f : V \to \mathbf{R} \mid f$ is linear $\}$

Proposition: With respect to this choice of basis, the matrix of $d_p f$: $T_p M \to T_q N$ is $(\frac{\partial F_i}{\partial x_j})_{\varphi(p)}$, where x_j are coordinates in $\mathbf{R}^m$ and $F_i = (F_1, \cdots, F_n)$ in coordinates for $\mathbf{R}^n$.

Here are some more properties of $d_p f$ and $T_p M$: Let $f\colon M \to N$ be smooth.

1.) If $f\colon M \to N$ is a diffeomorphism, the $d_p f$ is an isomorphism, for all $p \in M$.

2.) If $d_p f = 0$ for all $p \in M$ iff f is a constant map.

3.) If $id : M \to M$, $id(x) = x$, then $d_p(id) = I_m$

4.) $d_r(f \circ h) = d_p(f) \circ d_r(h)$ where $p = h(r)$.

5.) If $N \cong M/G$, where G is a discrete Lie group acting properly discontinuously on M, and $f\colon M \to M/G$ is the orbit map, the $d_p f$ is an isomorphism for all p.

6.) $T_{(p,q)}(M \times N) \cong T_p(M) \times T_q(N)$.

Rank Theorem: Suppose $A_0 \subset \mathbf{R}^n$, $B_0 \subset \mathbf{R}^m$, $F : A_0 \to B_0 \in C^1$ $a \in A_0, b \in B_0$. Suppose rank F $= k$.

Then there exists $A^{open} \subset A_0$ such that $a \in A$ and $B^{open} \subset B_0$ such that $b \in B$ and G, H, C^r diffeomorphisms

such that $G : A \to U^{open} \subset \mathbf{R}^n$, $H : B \to V^{open} \subset \mathbf{R}^m$ and

$$H \circ F \circ G^{-1}(x_1, ..., x_n) = (x_1, ..., x_k, 0, ..., 0)$$

Proposition: rank $DF(a) = k$ implies there exists V open such that $a \in V$ and $DF(x) \geq k$ for all $x \in V$

Proof:

Rank of $A = dim(span\{\mathbf{r}_1, ..., \mathbf{r}_m\}) = dim(span\{\mathbf{c}_1, ..., \mathbf{c}_m\})$
$=$ maximum order of any nonvanishing minor determinant.

Use determinant is a continuous function.

Thm 6.4 (Inverse Function Theorem): Suppose $F : W^{open} \subset \mathbf{R}^n \to \mathbf{R}^n \in C^r$. Suppose for $a \in W$, $det(DF_a) \neq 0$. Then there exists U such that $a \in U^{open}$, $V = F(U)$ is open, and $F : U \to V$ is a C^r-diffeomorphism. Moreover, for $x \in U$ and $y = F(x)$, $DF_y^{-1} = (DF_x)^{-1}$