Chapter 1 all sections

1.3

Defn: M is *locally Euclidean of dimension* n if for all $p \in M$, there exists an open set U_p such that $p \in U_p$ and there exists a homeomorphism $f_p : U_p \to V_p$ where $V_p \subset \mathbf{R}^n$.

(U_p, f) is a *coordinate nbhd* of p.

Let $q \in U \subset M$. $f(q) = (f_1(q), f_2(q), ..., f_n(q)) \in \mathbf{R}^n$ are the *coordinates* of q.

f_i is the *ith coordinate function*

Defn 3.1: An *n-manifold*, M, is a topological space with the following properties:

1.) M is locally Euclidean of dimension n.

2.) M is Hausdorff.

3.) M has a countable basis.

1.4

Let upper half-space, $H^n = \{(x_1, x_2, ..., x_n) \in \mathbf{R}^n \mid x_n \geq 0\}$,

$\partial H^n = \{(x_1, x_2, ..., x_{n-1}, 0) \in \mathbf{R}^n\} \sim \mathbf{R}^{n-1}$

M is a manifold with boundary if it is Hausdorff, has a countable basis, and if for all $p \in U$, there exists an open set U_p such that $p \in U_p$ and there exists a homeomorphism $f : U_p \to V_p$ one of the following holds:

i.) $V_p \subset H^n - \partial H^n$ (p is an interior point) or

ii.) $V_p \subset H^n$ and $f(p) \in \partial H^n$ (p is a boundary point).

∂M = set of all boundary points of M is an (n-1)-dimensional manifold.

1.5

Ex: $P^n(\mathbf{R}) = \mathbf{R}P^n = \mathbf{R}P^n = (\mathbf{R}^n - \{\mathbf{0}\})/(\mathbf{x} \sim t\mathbf{x})$
$= n$-dimensional real projective space.

$T(M) = \cup_{p \in M} T_p(M)$

Ch 2: all sections

2.1

Defn: Let $f : \mathbf{R}^n \to \mathbf{R}^m$. Define $g_{ij} : \mathbf{R}^1 \to \mathbf{R}^1$,
$g_{ij}(t) = f_i(x_1, ..., x_{j-1}, t, x_{j+1}, ..., x_n)$. If g is differentiable at a, then the partial derivative of f_i is defined by

$\frac{\partial f_i}{\partial x_j}(\mathbf{a}) = lim_{h\to 0}\frac{f_i(a_1,...,a_{j-1},a_j+h,a_{j+1},...,a_n)-f_i(a_1,...,a_{j-1},a_j,a_{j+1},...,a_n)}{h}$

$= lim_{h\to 0}\frac{f_i(\mathbf{a}+h\mathbf{e_j})-f_i(\mathbf{a})}{h}$

Defn: Suppose $A \subset \mathbf{R}^n$, $f : A \to \mathbf{R}^m$.

f is said to be **differentiable at a point a** if there exists an open ball V such that $a \in V \subset A$ and a linear function T such that

$$lim_{\mathbf{h}\to\mathbf{0}}\frac{||f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - T(\mathbf{h})||}{||\mathbf{h}||} = 0$$

$$lim_{\mathbf{x}\to\mathbf{a}}\frac{||f(\mathbf{x}) - f(\mathbf{a}) - T(\mathbf{x}-\mathbf{a})||}{||\mathbf{x}-\mathbf{a}||} = 0$$

OR equivalently,

f is differentiable at $\mathbf{a}$ if and only if there is a matrix T and a function $\epsilon : R^n \to R^m$ such that $lim_{\mathbf{h}\to 0}\epsilon(\mathbf{h}) = \mathbf{0}$ and

$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) = T(\mathbf{h}) + ||\mathbf{h}||\epsilon(\mathbf{h})$$

Or equivalently, there exists an m-tuple, $R(\mathbf{x},\mathbf{a}) = (r_1(\mathbf{x},\mathbf{a}), r_2(\mathbf{x},\mathbf{a}), ..., r_m(\mathbf{x},\mathbf{a})$
such that $lim_{\mathbf{x}\to a}||R(\mathbf{x},\mathbf{a})|| = 0$ and

$$f(\mathbf{x}) = f(\mathbf{a}) + T(\mathbf{x}-\mathbf{a}) + ||\mathbf{x}-\mathbf{a}||R(\mathbf{x},\mathbf{a})$$

Defn: Let V be a nonempty open subset of $\mathbf{R}^n$, $f : V \to \mathbf{R}^m$, $p \in \mathbf{N}$.
i.) f is C^p on V is each partial derivative of order $k \leq p$ exists and is continuous on V.

ii.) f is C^∞ (or *smooth*) on V if f is C^p on V for all $p \in \mathbf{N}$ (f is *smooth*).

iii.) f is C^ω (or *analytic*) on V if for all $a \in V$, near a, each component function can be written as a power series (e.g. if $f : \mathbf{R} \to \mathbf{R}^m$, $f(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+...$ (its Taylor series)).

2.2

Defn: The **Jacobian matrix of f at a** is

$$\left[\frac{\partial f_i}{\partial x_j}(\mathbf{a})\right]_{m\times n} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & ... & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ . & . & . \\ . & . & . \\ . & . & . \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & ... & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}$$

2.3

$T_{\mathbf{a}}(\mathbf{R}^n) = \{(\mathbf{a}, \mathbf{x}) \mid \mathbf{x} \in \mathbf{R}^n\}$, $\phi(\mathbf{ax}) = \mathbf{x} - \mathbf{a}$,

canonical basis $= \{\phi^{-1}(\mathbf{e}_i) \mid i = 1, ..., n\}$

Let $x(t) : \mathbf{R} \to \mathbf{R}^n$, a C^1 curve such that $x(0) = \mathbf{a}$

$T_{\mathbf{a}}(\mathbf{R}^n) = \{[x(t)] \mid x \in C^1, x(0) = \mathbf{a}\}$ where $x(t) \sim y(t)$ if $x_i'(t) = y_i'(t)$ for $t \in (-\epsilon, \epsilon)$

Let $f([x(t)]) = \mathbf{x}'(0) = (x_1'(0), ..., x_n'(0))$, then
$[x(t)] + [y(t)] = f^{-1}(x'(0) + y'(0))$ and $\alpha[x(t)] = f^{-1}(\alpha x'(0))$

Let $f : \mathbf{R}^n \to \mathbf{R}$ and let $\mathbf{v} \in R^n$ such that $||\mathbf{v}|| = 1$

The directional derivative of f at $\mathbf{a}$ in the direction of $\mathbf{v}$ is $D_{\mathbf{v}} f(\mathbf{a}) = lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{v})-f(\mathbf{a})}{h}$
$= D[f(\mathbf{a} + t\mathbf{v})]_0 = Df_{\mathbf{a}}\mathbf{v} = Df_{\mathbf{a}} \cdot \mathbf{v} = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})|_a \cdot \mathbf{v} = \nabla f \cdot v$

2.4

$C^\infty(a) = \{[f] : X \subset \mathbf{R}^n \to \mathbf{R} \in C^\infty \mid a \in domf\}$

where $f \sim g$ if $\exists U^{open}$ s.t. $\mathbf{a} \in U$ and $f(x) = g(x) \forall x \in U$.

If $X_{\mathbf{a}} = \Sigma_{i=1}^n \xi_i E_{i\mathbf{a}}$, then $X_{\mathbf{a}}^* : C^\infty(\mathbf{a}) \to \mathbf{R}$, $X_{\mathbf{a}}^*(f) = \Sigma_{i=1}^n \xi_i \frac{\partial f}{\partial x_i}|_{\mathbf{a}}$

Note $E_{i\mathbf{a}}^* = \frac{\partial}{\partial x_i}|_{\mathbf{a}}$

The directional derivative of f at $\mathbf{a}$ in the direction of $X_{\mathbf{a}} = X_{\mathbf{a}}^*(f)$.

Let $\mathcal{D}(a) = \{D : C^\infty(\mathbf{a}) \to \mathbf{R} \mid D$ is linear and satisfies the Leibniz rule $\}$

$D \in \mathcal{D}(a)$ is called a *derivation*

$j : T_{\mathbf{a}}(\mathbf{R}^n) \to \mathcal{D}(a)$, $j(X_{\mathbf{a}}) = X_{\mathbf{a}}^*$ is an isomorphism.

2.5

Defn: A *vector field* is a function, $\mathcal{V} : U \to \cup_{\mathbf{a} \in U} T_{\mathbf{a}}(\mathbf{R}^n)$, such that $\mathcal{V}(\mathbf{a}) \in T_{\mathbf{a}}(\mathbf{R}^n)$

Defn: A vector field is *smooth* if its components relative to the canonical basis $\{E_{i\mathbf{a}} \mid i = 1, ..., n\}$ are smooth.

Defn: A *field of frames* is a set of vector fields $\{\mathcal{V}_1, ..., \mathcal{V}_2\}$ such that $\{\mathcal{V}_1(\mathbf{a}), ..., \mathcal{V}_2(\mathbf{a})\}$ forms a basis for $T_{\mathbf{a}}(\mathbf{R}^n)$ for all $\mathbf{a}$.

Note we can turn a vector field into a derivation by making the following definition:

If $\mathcal{V}(\mathbf{a}) = \alpha_i(\mathbf{a}) E_{i\mathbf{a}}$, then $\mathcal{V} : C^\infty \to C^\infty$, $\mathcal{V}(f)(\mathbf{a}) = \Sigma_{i=1}^n \alpha_i(\mathbf{a}) \frac{\partial f}{\partial x_i}(\mathbf{a})$ is a derivation.

2.6

F is a C^r-*diffeomorphism* if

(1) F is a homeomorphism

(2) $F, F^{-1} \in C^r$

F is a *diffeomorphism* if F is a C^∞-diffeomorphism.

2.7

Rank of $A = dim(span\{\mathbf{r}_1, ..., \mathbf{r}_m\}) = dim(span\{\mathbf{c}_1, ..., \mathbf{c}_m\})$
= maximum order of any nonvanishing minor determinant.

Rank of $F : U \subset \mathbf{R}^n \to \mathbf{R}^m$ at x = rank of $DF(x)$

F has rank k if F has rank k at each x.

Chapter 3: 1 - 3

3.1 = Randell 1.1

Note: This is one place where Boothby's and Randell's definitions differ: Boothby's atlas = Randell's pre-atlas, Boothby's maximal atlas = Randell's atlas. On exams, I will use pre-atlas and maximal atlas when it makes a difference.

Defn: (φ, U) is a *chart* or *coordinate neighborhood* if $\varphi : U \to U'$ is a homeomorphism, where U is open in M and U' is open in $\mathbf{R}^n$.

(φ, U) is a *coordinate nbhd* of p.

Let $q \in U \subset M$. $\varphi(q) = (\varphi_1(q), \varphi_2(q), ..., \varphi_n(q)) \in \mathbf{R}^n$ are the *coordinates* of q.

φ_i is the *ith coordinate function.*

Defn: Two charts, (φ, U) and (ψ, V) are C^∞ *compatible* if the function $\varphi\psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is a diffeomorphism.

Defn: A *(pre) atlas* or *differentiable or smooth structure* on M is a collection of charts on M satisfying the following two conditions:

i.) the domains of the charts form an open cover of M

ii.) Each pair of charts in the atlas is compatible.

Defn: An atlas is a (maximal or complete) atlas if it is maximal with respect to properties i) and ii).

A *differential (or smooth or C^∞) n-manifold* M is a topological n-manifold together with a maximal atlas.

3.2

Let $\sim$ be an equivalence relation on X. Let $\pi : X \to X/\sim$, $\pi(x) = [x] = \{y \mid y \sim x\}$

$[A] = \cup_{a \in A}[a]$

Defn: $\sim$ is *open* if $U^{open} \subset X$ implies $[U]$ open in $X/\sim$.

3.3 = Randell 1.2

Defn: Suppose $f : W \to N$ where $W \subset M$ and N are smooth manifolds. f is *smooth* if for all $p \in W$, $\exists$ charts (ϕ, U) and (φ, V) and such that $p \in U$, $f(p) \in V$, $f(U) \subset V$ and $\varphi \circ f \circ \phi^{-1}$ is smooth.

Defn: $f : M \to N$ is a *diffeomorphism* if f is a homeomorphism and if f and f^{-1} are smooth. M and N are *diffeomorphic* if there exists a diffeomorphism $f : M \to N$.

Randell Chapter 1.3

Defn: G is a *topological group* if

1.) $(G, *)$ is a group
2.) G is a topological space.
3.) $* : G \times G \to G$, $*(g_1, g_2) = g_1 * g_2$, and
$In : G \to G$, $In(g) = g^{-1}$ are both continuous functions.

Defn: G is a *Lie group* if

1.) G is a topological group

2.) G is a smooth manifold.

3.) $*$ and In are smooth functions.

Defn: G = group, X = set. G *acts on* X (on the left) if $\exists \sigma : G \times X \to X$ such that

1.) $\sigma(e, x) = x \ \ \forall x \in X$

2.) $\sigma(g_1, \sigma(g_2, x)) = \sigma(g_1 g_2, x)$

Notation: $\sigma(g, x) = gx$.
Thus 1) $ex = x$; 2) $g_1(g_2 x) = (g_1 g_2)(x)$.

If G is a topological group and X is a topological space, then we require σ to be continuous. If G is a Lie group and X is a smooth manifold, then we require σ to be smooth.

Defn: The *orbit* of $x \in X$ =
$G(x) = \{y \in X \mid \exists g \text{ such that } y = gx\}$

Defn: If G acts on X, then $X/G = X/\sim$ where $x \sim y$ iff $y \in G(x)$ iff $\exists g$ such that $y = gx$.

Defn: The action of G on X is *free* if $gx = x$ implies $g = e$.

Defn: G is a *discrete group* if

0.) G is a group.

1.) G is countable

2.) G has the discrete topology

Defn: The action of G on M is *properly discontinuous* if $\forall x \in M$, $\exists U^{open}$ such that $x \in U$ and $U \cap gU = \emptyset \ \ \forall g \in G$.

Randell 2.1

Let $p \in M$.

A *germ* is an equivalence class [g] where
$g^{smooth} : U \to N$, for some U^{open} such that $p \in U \subset M$
and if $g_i : U_i \to N$, where $p \in U_i^{open} \subset M$,
then $g_1 \sim g_2$ if $\exists V^{open}$ such that $p \in V \subset U_1 \cap U_2$ and $g_1(x) = g_2(x)\ \forall x \in V$.

$G(p, N) = \{[g] \mid g^{smooth} : U \to N$, for some U^{open} such that $p \in U \subset M\}$

$G(p) = G(p, \mathbf{R})$

Let $\alpha : I \to M$ where $I =$ an interval $\subset \mathbf{R}$, $\alpha(0) = p$. Note $[\alpha] \in G[0, M]$

Directional derivative of $[g]$ in direction $[\alpha] =$

$$D_\alpha g = \frac{d(g \circ \alpha)}{dt}|_{t=0} \in \mathbf{R}$$

$T_p(M) = \{v : G(p) \to \mathbf{R} \mid v$ is linear and satisfies the Leibniz rule $\}$

$v \in T_p(M)$ is called a *derivation*

Given a chart (U, ϕ) at p where $\phi(p) = \mathbf{0}$, the *standard basis* for $T_p(M) = \{v_1, ..., v_m\}$, where $v_i = D_{\alpha_i}$ and for some $\epsilon > 0$, $\alpha_i : (-\epsilon, \epsilon) \to M$, $\alpha_i(t) = \phi^{-1}(0, ..., t, ..., 0)$

Suppose $f^{smooth} : M \to N$, $f(p) = q$. The *tangent (or differential) map*, $df_p : T_pM \to T_qN$, $(df_p(v)) = v([g \circ f])$ for $v \in T_pM$, $[g] \in G(q)$

I.e., df_p takes the derivation $(v : G(p) \to \mathbf{R}) \in T_pM$ to the derivation in T_qN which takes the germ $[g] \in G(q)$ to the real number $v([g \circ f])$.