Randell A1:

Defn: Suppose $f : M \to N$ where M and N are smooth manifolds. f is *smooth* if for all $p \in M$, $\exists$ charts (ϕ, U) and (φ, V) and such that $p \in U$, $f(p) \in V$, $f(U) \subset V$ and $\varphi \circ f \circ \phi^{-1}$ is smooth.

Note this definition is equivalent to:

Suppose $f : M \to N$ where M and N are smooth manifolds. f is *smooth* if for all $p \in M$ and for all charts (ϕ, U) and (φ, V) such that $p \in U$, $f(p) \in V$, $f(U) \subset V$, then $\varphi \circ f \circ \phi^{-1}$ is smooth.

Suppose $g : M \to W$ and $f : W \to N$ are smooth. Let M be an m-manifold, W a k-manifold, and N an n-manifold.

Claim $f \circ g : M \to N$ is smooth.

Let $p \in M$. g smooth implies $\exists$ charts (ϕ_1, U_1) and (φ_1, V_1) such that $p \in U_1$, $\phi_1(p) \in V_1$, $g(U_1) \subset V_1$ and $\varphi_1 \circ g \circ \phi_1^{-1} : \phi(U_1) \subset R^m \to \varphi_1(V_1) \subset R^k$ is smooth.

Let (φ_2, V_2) be a chart such that $f(g(p)) \in V_2$. Let $V_3 = f|_W^{-1}(V_2) \cap V_1$. Then $g(p) \in V_3$ and $f(V_3) \subset V_2$

f smooth implies f is continuous. Thus V_3 is open in W and $(\varphi_1|_{V_3}, V_3)$ is a chart.

f smooth implies $\varphi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(V_3) \subset R^k \to \varphi_2(V_2) \subset R^n$ is smooth.

Thus $(\varphi_2 \circ f \circ \varphi_1^{-1}) \circ (\varphi_1 \circ g \circ \phi_1^{-1}) = \varphi_2 \circ f \circ g \circ \phi_1^{-1} : \phi(U_1) \subset R^m \to \varphi_2(V_2) \subset R^n$ is smooth.

Thus $f \circ g$ is smooth

A2

$T_p(M) = \{v : G(p) \to \mathbf{R} \mid v \text{ is linear and satisfies the Leibniz rule } \}$

Given a chart (U, ϕ) at p where $\phi(p) = \mathbf{0}$, the *standard basis* for $T_p(M) = \{v_1, ..., v_m\}$, where $v_i = D_{\alpha_i}$ and for some $\epsilon > 0$, $\alpha_i : (-\epsilon, \epsilon) \to M$, $\alpha_i(t) = \phi^{-1}(0, ..., t, ..., 0)$

B1) Let $U = \Gamma(f)$, $\phi : \Gamma(f) \to R^2$, $\phi(x, y, z) = (x, y)$.

Since the domain of f is R^2, $\Gamma(f)$ is onto.

$\Gamma(f) \subset \Gamma(f)$, $\Gamma(f)$ is open in $\Gamma(f)$, and R^2 is open in R^2, thus if ϕ is a homeomorphism, $\{(\phi, \Gamma(f))\}$ is a pre-atlas.

Suppose $\phi(x_1, y_1, z_1) = \phi(x_2, y_2, z_2)$. Then $(x_1, y_1) = \phi(x_1, y_1, z_1) = \phi(x_2, y_2, z_2) = (x_2, x_2)$. Also $z_1 = f(x_1, y_1) = f(x_2, x_2) = z_2$. Thus $x_1 = x_2, y_1 = y_2, z_1 = z_2$. Hence ϕ is 1:1.

Let $\pi_{xy} : \mathbf{R}^3 \to \mathbf{R}^2$, $\pi_{xy}(x, y, z) = (x, y)$. If U is open in $\mathbf{R}^2$, $\pi_{xy}^{-1}(U) = U \times \mathbf{R}$ which is open in $\mathbf{R}^3$. Thus π_{xy} is continuous and $\phi = \pi_{xy}|_{\Gamma(f)}$ is continuous.

Let V be open in $\mathbf{R}^2$. Then $\pi_{xy}|_{\Gamma(f)}(V) = (V \times \mathbf{R}) \cap \Gamma(f)$ is open in $\Gamma(f)$.

Thus $\phi : \Gamma(f) \to R^2$ is a homeomorphism.

Since $\Gamma(f)$ has a pre-atlas, $\Gamma(f)$ is a smooth manifold.