Defn: Suppose $f : M \to N$ where $M$ and $N$ are smooth manifolds. $f$ is smooth if for all $p \in M$, $\exists$ charts $(\phi, U)$ and $(\varphi, V)$ and such that $p \in U$, $f(p) \in V$, $f(U) \subset V$ and $\varphi \circ f \circ \phi^{-1}$ is smooth.

Note this definition is equivalent to:

Suppose $f : M \to N$ where $M$ and $N$ are smooth manifolds. $f$ is smooth if for all $p \in M$ and for all charts $(\phi, U)$ and $(\varphi, V)$ such that $p \in U$, $f(p) \in V$, $f(U) \subset V$, then $\varphi \circ f \circ \phi^{-1}$ is smooth.

Suppose $g : M \to W$ and $f : W \to N$ are smooth. Let $M$ be an $m$-manifold, $W$ a $k$-manifold, and $N$ an $n$-manifold.

Claim $f \circ g : M \to N$ is smooth.

Let $p \in M$. $g$ smooth implies $\exists$ charts $(\phi_1, U_1)$ and $(\varphi_1, V_1)$ such that $p \in U_1$, $\phi_1(p) \in V_1$, $g(U_1) \subset V_1$ and $\varphi_1 \circ g \circ \phi_1^{-1} : \phi(U_1) \subset R^m \to \varphi_1(V_1) \subset R^k$ is smooth.

Let $(\varphi_2, V_2)$ be a chart such that $f(g(p)) \in V_2$. Let $V_3 = f|_W^{-1}(V_2) \cap V_1$. Then $g(p) \in V_3$ and $f(V_3) \subset V_2$.

$f$ smooth implies $f$ is continuous. Thus $V_3$ is open in $W$ and $(\varphi_1|_{V_3}, V_3)$ is a chart.

$f$ smooth implies $\varphi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(V_3) \subset R^k \to \varphi_2(V_2) \subset R^n$ is smooth.

Thus $(\varphi_2 \circ f \circ \varphi_1^{-1}) \circ (\varphi_1 \circ g \circ \phi_1^{-1}) = \varphi_2 \circ f \circ g \circ \phi_1^{-1} : \phi(U_1) \subset R^m \to \varphi_2(V_2) \subset R^n$ is smooth.

Thus $f \circ g$ is smooth.

A2

$T_p(M) = \{v : G(p) \to \mathbb{R} \mid v$ is linear and satisfies the Leibniz rule $\}$

Given a chart $(U, \phi)$ at $p$ where $\phi(p) = 0$, the standard basis for $T_p(M) = \{v_1, ..., v_m\}$, where $v_i = D_{\alpha_i}$ and for some $\epsilon > 0$, $\alpha_i : (-\epsilon, \epsilon) \to M$, $\alpha_i(t) = \phi^{-1}(0, ..., t, ..., 0)$
B1) Let $U = \Gamma(f), \phi : \Gamma(f) \to \mathbb{R}^2, \phi(x, y, z) = (x, y)$.

Since the domain of $f$ is $\mathbb{R}^2$, $\Gamma(f)$ is onto.

$\Gamma(f) \subset \Gamma(f)$, $\Gamma(f)$ is open in $\Gamma(f)$, and $\mathbb{R}^2$ is open in $\mathbb{R}^2$, thus if $\phi$ is a homeomorphism, $\{(\phi, \Gamma(f))\}$ is a pre-atlas.

Suppose $\phi(x_1, y_1, z_1) = \phi(x_2, y_2, z_2)$. Then $(x_1, y_1) = \phi(x_1, y_1, z_1) = \phi(x_2, y_2, z_2) = (x_2, x_2)$. Also $z_1 = f(x_1, y_1) = f(x_2, x_2) = z_2$. Thus $x_1 = x_2, y_1 = y_2, z_1 = z_2$. Hence $\phi$ is 1:1.

Let $\pi_{xy} : \mathbb{R}^3 \to \mathbb{R}^2, \pi_{xy}(x, y, z) = (x, y)$. If $U$ is open in $\mathbb{R}^2$, $\pi_{xy}^{-1}(U) = U \times \mathbb{R}$ which is open in $\mathbb{R}^3$. Thus $\pi_{xy}$ is continuous and $\phi = \pi_{xy}|_{\Gamma(f)}$ is continuous.

Let $V$ be open in $\mathbb{R}^2$. Then $\pi_{xy}|_{\Gamma(f)}(V) = (V \times \mathbb{R}) \cap \Gamma(f)$ is open in $\Gamma(f)$.

Thus $\phi : \Gamma(f) \to \mathbb{R}^2$ is a homeomorphism.

Since $\Gamma(f)$ has a pre-atlas, $\Gamma(f)$ is a smooth manifold.