Let V be a finite-dimensional vector space over $\mathbf{R}$.

The dual of $V = V^* = \{f : V \to \mathbf{R} \mid f \text{ linear } \}$

Note V^* is a vector space. The elements of V^* are called *covectors*.

If $e_1, ..., e_n$ basis for V, then $w_1, ..., w_n$ basis for V^* where $w_i : V \to \mathbf{R}$ where $w_i(e_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Let $F_* : V \to W$ be a linear map between vector spaces
The dual map map is $F^* : W^* \to V^*$, $F(g) = g \circ F$.

F_* is injective implies F^* injective

F_* is surjective implies F^* surjective

$(G_* \circ F_*)^* = F^* \circ G^*$.

$d : V \to (V^*)^*$, $d(v) = h$ where $h : V^* \to R$, $h(f) = f(v)$.

Thus $(V^*)^*$ is naturally isomorphic to V.

Defn: The dual of $T_pM = T_p^*M$ is the *cotangent space* to M at p.

If $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}$ is a basis for T_pM, then the dual basis will be denoted $dx_1, ..., dx_m$.

B is bilinear if
$B(cv_1 + dv_2, w) = cB(v_1, w) + dB(v_2, w)$
$B(v, cw_1 + dw_2) = cB(v, w_1) + dB(v, w_2)$

Thus

$B((v_1, w_1) + (v_2, w_2)) = B(v_1 + v_2, w_1 + w_2)$
$= B(v_1, w_1 + w_2) + B(v_2, w_1 + w_2)$
$= B(v_1, w_1) + B(v_1, w_2) + B(v_2, w_1) + B(v_2, w_2)$

B is linear if
$B((v_1, w_1) + (v_2, w_2)) = B((v_1, w_1)) + B((v_2, w_2))$
$B(c(v_1, w_1)) = cB((v_1, w_1)$

Let $(V \times W)^b = \{B : V \times W \to \mathbf{R} \mid B \text{ bilinear } \}$

$V \otimes W = [(V \times W)^b]^* = \{h : (V \times W)^b \to \mathbf{R} \mid h \text{ linear } \}$

If $v \in V, w \in W$, let $v \otimes w : (V \times W)^b \to \mathbf{R},\ (v \otimes w)(B) = B(v, w)$

Prop: $v \otimes w \in V \otimes W$.

Proof: $(v \otimes w)(c_1B_1 + c_2B_2) = (c_1B_1 + c_2B_2)(v, w) = c_1B_1(v, w) + c_2B_2(v, w) = c_1(v \otimes w)(B_1) + c_2(v \otimes w)(B_2)$.

Prop: $\Sigma_{i=1}^n r_i(v_i \otimes w_i) \in V \otimes W$.

Proof: $(\Sigma_{i=1}^n r_i(v_i \otimes w_i))(c_1B_1 + c_2B_2) = \Sigma_{i=1}^n r_i(v_i \otimes w_i)(c_1B_1 + c_2B_2) = \Sigma_{i=1}^n c_1r_i(v_i \otimes w_i)(B_1) + c_2r_i(v_i \otimes w_i)(B_2) = c_1(\Sigma_{i=1}^n r_i(v_i \otimes w_i))(B_1) + c_2(\Sigma_{i=1}^n r_i(v_i \otimes w_i))(B_2)$.

Note $(\Sigma_{i=1}^n r_i(v_i \otimes w_i))(B) = \Sigma_{i=1}^n r_i(v_i \otimes w_i)(B) = \Sigma_{i=1}^n r_iB(v_i, w_i)$

Prop: The product $\otimes$ is bilinear:

Proof: $((c_1v_1 + c_2v_2) \otimes w)(B) = B(c_1v_1 + c_2v_2, w) = c_1B(v_1, w) + c_2B(v_2, w)$
and ...

Claim: $V \otimes W =< v \otimes w \mid v \in V, w \in W,$
$(c_1v_1 + c_2v_2) \otimes w = c_1(v_1 \otimes w) + c_2(v_2 \otimes w),$
$v \otimes (c_1w_1 + c_2w_2) = c_1(v \otimes w_1) + c_2(v \otimes w_2) >$

$=< v \otimes w \mid v \in V, w \in W,\ (v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w,$
$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2, (cv) \otimes w = c(v \otimes w) = v \otimes (cw) >$

Let $v_1, ..., v_m$ be a basis for V, $w_1, ..., w_n$ basis for W. If B is a bilinear form, then B is uniquely determined by the nm values $B(v_i, w_j)$.

Thus $dim(V \times W)^b = nm = V \otimes W$.

Hence a basis for $V \otimes W$ is $\{v_i \otimes w_j \mid i = 1, ..., m, j = 1, ..., n\}$

$$\text{Thus } V \otimes W = \{\Sigma c_{ij} v_i \otimes w_j \mid \ (v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w,$$
$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2, (cv) \otimes w = c(v \otimes w) = v \otimes (cw)\}$$

The universal mapping property: There exists ϕ, $V \otimes W$ such that ϕ is bilinear and given any bilinear map: $B : V \times W \to U$, there exists a map $A : V \otimes W \to U$ such that $A \circ \phi = B$. Moreover ϕ and $V \otimes W$ are unique in the sense that if X and ψ satisfy the universal mapping property, then there exists an isomorphism $f : V \otimes W \to X$ such that $f \circ \phi = \psi$

Fix $B : V \times W \to U$ and take $f : U \to \mathbf{R} \in U^*$

Then $f \circ B : V \times W \to \mathbf{R}$ is bilinear:

$(f \circ B)(c_1 v_1 + c_2 v_2, w) = f(c_1 B(v_1, w) + c_2 B(v_2, w)) = c_1 (f \circ B)(v_1, w) + c_2 (f \circ B)(v_2, w)$

Similarly $(f \circ B)(v, c_1 w_1 + c_2 w_2) = c_1 (f \circ B)(v, w_1) + c_2 (f \circ B_(v, w_2)$

Thus $f \circ B \in (V \times W)^b$

$\alpha : U^* \to (V \times W)^b$, $\alpha(f) = f \circ B$

Thus $\beta : V \otimes W \to (U^*)^*$, β

The tensor algebra $= T(V) = \oplus_{k=0}^{\infty} (\otimes^k V)$

$c_0 + c_1 v_i + \Sigma c_{ij} v_i \otimes v_j + ... + \Sigma c_{i_1 ... i_p} v_{i_1} \otimes ... \otimes v_{i_p}$

$(v_1 \otimes ... \otimes v_p)(u_1 \otimes ... \otimes u_q) = v_1 \otimes ... \otimes v_p \otimes u_1 \otimes ... \otimes u_q$

A subring I of a ring R is an *ideal* of R if $ar \in I$ and $ra \in I$ for all $a \in I$, $r \in R$.

Let $I(V) =$ ideal generated by $\{v \otimes v \mid v \in V\}$

The exterior algebra of V is the quotient $\Lambda^* V = T(V)/I(V)$
