Defn: $G$ is a topological group if
1.) $(G, \ast)$ is a group
2.) $G$ is a topological space.
3.) $\ast : G \times G \to G$, $\ast(g_1, g_2) = g_1 \ast g_2$, and $In : G \to G$, $In(g) = g^{-1}$ are both continuous functions.

Defn: $G$ is a Lie group if
1.) $G$ is a topological group
2.) $G$ is a smooth manifold.
3.) $\ast$ and $In$ are smooth functions.

Ex: $(\mathbb{R}, +)$, $(\mathbb{R} - \{0\}, \cdot)$, $(\mathbb{C} - \{0\}, \cdot)$, $(S^1, \cdot)$ where $S^1 \subset \mathbb{C}$, $(\mathbb{Z}, +)$, $(\mathbb{Z}_p, +)$, $(Gl(n, \mathbb{R}), \text{matrix multiplication})$ are Lie groups. For $G_1, G_2$ lie groups, $G_1 \times G_2$ is a lie group.

Defn: $G = \text{group}, X = \text{set}$. $G$ acts on $X$ (on the left) if $\exists \sigma : G \times X \to X$ such that
1.) $\sigma(e, x) = x \ \forall x \in X$
2.) $\sigma(g_1, \sigma(g_2, x)) = \sigma(g_1g_2, x)$

Notation: $\sigma(g, x) = gx$.
Thus 1) $ex = x$; 2) $g_1(g_2x) = (g_1g_2)(x)$.

If $G$ is a topological group and $X$ is a topological space, then we require $\sigma$ to be continuous.

If $G$ is a Lie group and $X$ is a smooth manifold, then we require $\sigma$ to be smooth.

Defn: The orbit of $x \in X = G(x) = \{y \in X \mid \exists g \text{ such that } y = gx\}$

Note:
1.) $x \in G(x)$
2.) If $G(x) \cap G(y) \neq \emptyset$, then $G(x) = G(y)$

Thus we can use an action of $G$ to partition $X$ into disjoint subsets.

Hence the action of $G$ on $X$ can be used to define an equivalence relation on $X$: $x \sim y$ iff $y \in G(x)$ iff $\exists g$ such that $y = gx$.

$X/G = X/\sim$.

If $X$ is a topological space, then $X/G = X/\sim$ is a topological space with the quotient topology.

When is $X/G = X/\sim$ a manifold?

Ex: $G = (\mathbb{Z}, +), M = \mathbb{R}, \sigma(n, x) = n + x$.

$M/G =$

Ex: $G = (\mathbb{Z} \times \mathbb{Z}, +), M = \mathbb{R}^2,$
$\sigma((n, m), (x, y)) = (n + x, m + y)$.

$M/G =$

Ex: $G = (\mathbb{Z}_2, +), M = S^n, \sigma(0, x) = x, \sigma(1, x) = -x$.

$M/G =$
Defn: The action of $G$ on $X$ is free if $gx = x$ implies $g = e$.

Thm 1.3.9: If $M$ is a smooth $n$-manifold, and $G$ is a finite Lie group acting freely on $M$, then $M/G$ is a smooth $n$-manifold. Also, $p : M \to M/G$ is smooth.

Cor:

Defn: $G$ is a discrete group if

0.) $G$ is a group.

1.) $G$ is countable

2.) $G$ has the discrete topology

Note a discrete group is a Lie group.

Defn: The action of $G$ on $M$ is properly discontinuous if

$\forall x \in M, \exists U^{\text{open}}$ such that $x \in U$ and $U \cap gU = \emptyset \ \forall g \in G$.

Ex: $(\mathbb{Z}, +)$ acting on $\mathbb{R}^1$ where $\sigma(n, x) = n + x$.

Thm 1.3.2: $M$ smooth $n$-manifold, $G$ discrete group acting properly discontinuously on $M$ implies $M/G$ is a smooth $n$-manifold. Also, $p : M \to M/G$ is smooth.