HW 3:

Let \( F : \mathbb{R} \to \mathbb{R}^2, F(x) = (2,3)x \)
\( G : \mathbb{R}^2 \to \mathbb{R}^3, G(x,y) = (xy, x^2, x + 2y + 5) \)
\( H : \mathbb{R} \to \mathbb{R}^2, H(x) = (x^2, x^3) \)
\( k : \mathbb{R}^2 \to \mathbb{R}, k(x,y) = x^8 + 5xy. \)

1. Use the chain rule to calculate \( D(G \circ F)_2 \)

2. Use the product rule to calculate \( D(FH)_2 \)

3. Let \( \mathbf{a} = (3,4) \). Let \( X_{\mathbf{a}} = 9E_{1\mathbf{a}} - E_{2\mathbf{a}} \). Then \( X_{\mathbf{a}}(k) = \)

   \( F \) is a \( C^r \)-diffeomorphism if

   (1) \( F \) is a homeomorphism

   (2) \( F, F^{-1} \in C^r \)

   \( F \) is a diffeomorphism if \( F \) is a \( C^\infty \)-diffeomorphism.

4. Give an example of a homeomorphism which is analytic (ie \( C^\infty \) and near \( \mathbf{a} \), \( f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots \), its Taylor series) which is not a diffeomorphism.

5. Suppose \( F'(\mathbf{x}) = \mathbf{0} \) for all \( \mathbf{x} \in U^{open} \subset \mathbb{R}^n \). Show \( F \) cannot be a homeomorphism. What can you say about \( F \) (hint: MVT).

6. Suppose \( f : \mathbb{R} \to \mathbb{R}, f'(x) \neq 0 \) for all \( x \in U \). Show that the derivative of \( f^{-1} \) exists for all \( y \in f(U) \)

7. Suppose \( F \) is a \( C^1 \)-diffeomorphism. Show that \( DF_x \) is nonsingular (ie \( \det(DF_x) \neq 0) \forall x \in \text{dom}(F) \)

8. Ex 1: Show \( F : \mathbb{R}^n \to \mathbb{R}^n, F(\mathbf{x}) = \mathbf{x} + \mathbf{a} \) is a diffeomorphism.

9. Ex 2: Determine when \( F : \mathbb{R}^n \to \mathbb{R}^m, F(\mathbf{x}) = A\mathbf{x}, \) where \( A \) is an \( m \times n \) matrix, is a diffeomorphism. \( DF_x = \).

Note that if \( F \) and \( G \) are diffeomorphism, then \( F \circ G \) is a diffeomorphism (when \( F \circ G \) is defined).

Thm 6.5 (Contracting mapping theorem): Let \( M \) be a complete metric space and let \( T : M \to M \).
Suppose there exists a constant \( \lambda \in [0,1) \) such that for all \( x, y \in M, d(T(x), T(y)) \leq \lambda d(x,y) \).
Then \( T \) has a unique fixed point.

Proof: See class notes (Recall \( T^n(x_0) \) is a Cauchy sequence. Since \( M \) be a complete metric space, \( T^n(x_0) \) converges, say to \( a \). Then \( d(T(a), a) = 0) \).

Thm 6.4 (Inverse Function Theorem): Suppose \( F : W^{open} \subset \mathbb{R}^n \to \mathbb{R}^n \in C^r \). Suppose for \( a \in W, \)
\( \det(DF_a) \neq 0. \) Then there exists \( U \) such that \( a \in U^{open}, V = F(U) \) is open, and \( F : U \to V \) is a \( C^r \)-diffeomorphism. Moreover, for \( x \in U \) and \( y = F(x), DF_y^{-1} = (DF_x)^{-1} \)