2.5

Let $U \subset R^n$

Defn: A *vector field* is a function, $\mathcal{V} : U \to \cup_{\mathbf{a}\in U} T_{\mathbf{a}}(\mathbf{R}^n)$, such that $\mathcal{V} : (\mathbf{a}) \in T_{\mathbf{a}}(\mathbf{R}^n)$

Defn: A vector field is *smooth* if its components relative to the canonical basis $\{E_{i\mathbf{a}} \mid i = 1, ..., n\}$ are smooth.

Ex: $\mathcal{V} : \mathbf{R}^2 \to \cup_{\mathbf{a}\in U} T_{\mathbf{a}}(\mathbf{R}^2)$,

$\mathcal{V}(x, y) = (2x, -y) = 2xE_{1\mathbf{a}} - yE_{2\mathbf{a}}$.

Defn: A field of frames is a set of vector fields $\{\mathcal{V}_1, ..., \mathcal{V}_2\}$ such that $\{\mathcal{V}_1(\mathbf{a}), ..., \mathcal{V}_2(\mathbf{a})\}$ forms a basis for $T_{\mathbf{a}}(\mathbf{R}^n)$ for all $\mathbf{a}$.

Ex: $\{E_{1\mathbf{a}}, ..., E_{n\mathbf{a}}\}$ on $\mathbf{R}^n$.

Ex: $\{xE_{1\mathbf{a}} + yE_{2\mathbf{a}}, yE_{1\mathbf{a}} - xE_{2\mathbf{a}}\}$ on $\mathbf{R}^n - \{\mathbf{0}\}$.

Let $\mathcal{V} = \Sigma_{i=1}^n \alpha_i(\mathbf{a}) E_{i\mathbf{a}}$ be a smooth vector field on U.

$\mathcal{V} : C^\infty \to C^\infty$

$\mathcal{V}(f) = \Sigma_{i=1}^n \alpha_i(\mathbf{a}) \frac{\partial f}{\partial x_i}(\mathbf{a})$ is a derivation.

Thm 5.1: Let $F^{closed} \subset \mathbf{R}^n$, $K^{compact} \subset \mathbf{R}^n$, $F \cap K = \emptyset$. There there is a C^∞ function $\sigma : \mathbf{R}^n \to [0,1]$ such that $\sigma(K) = \{1\}$, $\sigma(F) = \{0\}$

Show that $h(t) = \begin{cases} 0 & t \leq 0 \\ e^{\frac{-1}{t}} & t > 0 \end{cases}$ is C^∞, (but not C^ω).

Let $\overline{g}(x) = \frac{h(\epsilon^2 - ||\mathbf{x}||^2)}{h(\epsilon^2 - ||\mathbf{x}||^2) + h(||\mathbf{x}||^2 - \frac{\epsilon^2}{4})}$

$$\overline{g}(\mathbf{x}) = \begin{cases} 1 & 0 \leq ||x|| \leq \frac{\epsilon}{2} \\ positive & \frac{\epsilon}{2} \leq ||x|| < \epsilon \\ 0 & ||x|| \geq \epsilon \end{cases}$$

Let $g(\mathbf{x}) = \overline{g}(\mathbf{x} - \mathbf{a})$

$$\sigma(\mathbf{x}) = 1 - \Pi_{i=1}^{k}(1 - g_i)$$

where $K \subset \cup_{i=1}^{k} B_{\frac{\epsilon}{2}}(\mathbf{a}_i)$ and $B_\epsilon(\mathbf{a}_i) \subset \mathbf{R}^n - F$.