\[ T_a(\mathbb{R}^n) = \{(a, x) \mid x \in \mathbb{R}^n\} \]

\[ \phi(ax) = x - a \]

canonical basis \( \{ E_{ia} = \phi^{-1}(e_i) \mid i = 1, \ldots, n\} \)

Let \( C^\infty(a) = \{ f : X \subset \mathbb{R}^n \to \mathbb{R} \in C^\infty \mid a \in \text{dom} f \} \)

\( f \sim g \) if \( \exists U^{\text{open}} \) s.t. \( a \in U \) and \( f(x) = g(x) \forall x \in U \).

\( f_i : U_i \to \mathbb{R} \in C^\infty(a) \) implies \( f_1 + f_2 : U_1 \cap U_2 \to \mathbb{R} \in C^\infty(a) \)

and \( \alpha f_i : U_i \to \mathbb{R} \in C^\infty(a) \)

Thus \( C^\infty(a) \) is an algebra over \( \mathbb{R} \)

Let \( X_a = \sum_{i=1}^{n} \xi_i E_{ia} \)

\( X^*_a : C^\infty(a) \to \mathbb{R} \)

\( X^*_a(f) = \sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_{ia}} \)

Let \( x_j : \mathbb{R}^n \to \mathbb{R} \), \( x_j(x) = x_j \)

\( X^*_a(x_j) = \sum_{i=1}^{n} \xi_i \frac{\partial x_j}{\partial x_{ia}} = \xi_i \)

\( X^*_a \) is linear and satisfies the Leibniz rule.

Let \( \mathcal{D}(a) = \{ D : C^\infty(a) \to \mathbb{R} \mid D \) is linear and satisfies the Leibniz rule \} \)

Define \( (\alpha D_1 + \beta D_2)(f) = \alpha[D_1(f)] + \beta[D_2(f)] \)
$\mathcal{D}(a)$ is closed under addition and scalar multiplication and hence is a vector space over $\mathbb{R}$

Let $j : T_a(\mathbb{R}^n) \to \mathcal{D}(a)$, $j(X_a) = X_a^*$

Claim: $j$ is an isomorphism.

Let $X_a = \sum_{i=1}^{n} \xi_i E_i a$ and $Z_a = \sum_{i=1}^{n} \zeta_i E_i a$

$j$ is a homomorphism.

$j$ is 1-1:

If $j(X_a) = j(Z_a)$, then $X_a^*(x_j) = \sum_{i=1}^{n} \xi_i \frac{\partial x_i}{\partial x_j} a = \xi_i = \zeta_i = Z_a^*(x_j)$

$j$ is onto:

Let $D$ be a derivation.

Suppose $f(x) = 1$. Then $Df = 0$

Suppose $g(x) = c$. Then $Dg = D(cf) = cDf = 0$

Let $h_i(x) = x_i$. Let $\xi_i = Dh_i$. Then $D = X_a^*$ where $X_a = \sum_{i=1}^{n} \xi_i E_i a$ (proof: long calculation, see Boothby).

Note since $X_a^*(f) = \sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i} a$, $j(E_i a) =$