Thm 2.3 (Chain rule): Suppose $U \subset \mathbb{R}^m$ is open and $f : U \to V \subset \mathbb{R}^m$, $g : V \to \mathbb{R}^p$. Let $h = g \circ f$. Suppose $f$ is differentiable at $a \in U$ and $g$ is differentiable at $f(a) \in V$. Then $h$ is differentiable at $a \in U$ and $D(h)_a = D(G)_{f(a)}D(f)_a$.

Let $R_h(x, a) = \frac{g(f(x)) - g(f(a)) - D(G)_{f(a)}D(f)_a(x-a)}{||x-a||}$

Let $y = f(x)$, $b = f(a)$

$$R_g(y, b) = \frac{g(y) - g(b) - D(G)_b(y-b)}{||y-b||}$$

where $\lim_{x \to a} R_g(y, b) = 0$

$$R_f(x, a) = \frac{f(x) - f(a) - D(f)_a(x-a)}{||x-a||}$$

where $\lim_{x \to a} R_f(x, a) = 0$

$$y - b = f(x) - f(a) = D(f)_a(x - a) + ||x - a||R_f(x, a)$$

$$R_g(y, b) = \frac{g(f(x)) - g(f(a)) - D(G)_bD(f)_a(x-a) + ||x-a||R_f(x, a)}{||y-b||}$$

$$\frac{||y-b||R_g(y, b)}{||x-a||} = \frac{g(f(x)) - g(f(a)) - D(G)_bD(f)_a(x-a) - D(G)_b||x-a||R_f(x, a)}{||x-a||}$$

$$\frac{||y-b||R_g(y, b) + D(G)_b||x-a||R_f(x, a)}{||x-a||} = \frac{g(f(x)) - g(f(a)) - D(G)_bD(f)_a(x-a)}{||x-a||}$$

$$R_h(x, a) = \frac{||y-b||R_g(y, b) + D(G)_b||x-a||R_f(x, a)}{||x-a||}$$

$$R_h(x, a) = \frac{||f(x) - f(a)||R_g(y, b)}{||x-a||} + D(G)_bR_f(x, a)$$

$$R_h(x, a) = \frac{||D(f)_a(x-a) + ||x-a||R_f(x, a)||R_g(y, b)}{||x-a||} + D(G)_bR_f(x, a)$$

Cor 2.4: If $f, g \in C^r$ on $U, V$ respectively, then $h = g \circ f \in C^r$.

Proof by induction:

$r = 1$: Suppose $f, g \in C^1$ on $U, V$ respectively.

Then $\frac{\partial f}{\partial x_i}$ exists and is continuous on $U$.

Then $\frac{\partial g}{\partial x_i}$ exists and is continuous on $V$.

By Thm 1.3, $f, g$ are differentiable on $U, V$ respectively.

Hence by Thm 1.1, $f$ is continuous. By Thm 2.3 $h = g \circ f$ is differentiable.

Thus by Thm 1.1, $\frac{\partial h}{\partial x_i}$ exist.

By Thm 2.3 it’s Jacobian is $D(h)_x = D(g)_{f(x)}D(f)_x$.

Since $\frac{\partial f_i}{\partial x_j}$ are continuous on $U$ for all $i, j$, each entry of $D(f)_x = (\frac{\partial f_i}{\partial x_j})$ is continuous.

Since $\frac{\partial g_i}{\partial x_j}$ are continuous on $V$ for all $i, j$ and $f$ is continuous, each entry of $D(g)_{f(x)} = (\frac{\partial g_i}{\partial x_j}|_{f(x)})$ is continuous.

Since the sums and products of continuous functions are continuous, each entry of $D(h)_a = D(G)_{f(a)}D(f)_a$ is continuous.