HW 2.1: 2, 8 (due Friday, next week)

\( f : \mathbb{R}^n \to \mathbb{R} \) is differentiable at \( a \) if

\[
\lim_{x \to a} \frac{f(x) - f(a) - T(x - a)}{||x - a||} = 0
\]

\[
\lim_{x \to a} \frac{f(x) - f(a) - \sum b_i(x_i - a_i)}{||x - a||} = 0
\]

\( y = f(a) + \sum b_i(x_i - a_i) \) approximates \( y = f(x) \)

\[
f(x) = f(a) + T(x - a) + ||x - a||r(x, a)
\]

where \( \lim_{x \to a} r(x, a) = 0 \)

Thm 1.1: If \( f \) is differentiable at \( a \), then

1.) \( f \) is continuous at \( a \).

2.) All partial derivatives exist at \( a \).

3.) \( b_i = \frac{\partial f}{\partial x_i} \) at \( a \)

Proof: 1.) \( \lim_{x \to a} f(x) = \lim_{x \to a} f(a) + T(x - a) + ||x - a||r(x, a) = f(a) \)

2,3.) \( \frac{\partial f}{\partial x_j}(a) = \lim_{h \to 0} \frac{f(a + h\mathbf{e}_j) - f(a)}{h} \)

\[
= \lim_{h \to 0} \frac{f(a) + T(a + h\mathbf{e}_j - a) + ||a + h\mathbf{e}_j - a||r(a + h\mathbf{e}_j, a) - f(a)}{h}
\]

\[
= \lim_{h \to 0} \frac{T(h\mathbf{e}_j) + ||r(a + h\mathbf{e}_j, a)||}{h} = \lim_{h \to 0} \frac{hT(e_j) + ||r(a + h\mathbf{e}_j, a)||}{h}
\]

Thm 1.3: If \( \frac{\partial f}{\partial x_j} \) exist for all \( j \) in a nbhd of \( a \) and if they are continuous at \( a \), then \( f \) is differentiable at \( a \).

Defn: Let \( V \) be a nonempty open subset of \( \mathbb{R}^n \), \( f : V \to \mathbb{R}^m \), \( p \in \mathbb{N} \).

i.) \( f \) is \( C^p \) on \( V \) is each partial derivative of order \( k \leq p \) exists and is continuous on \( V \).

ii.) \( f \) is \( C^\infty \) on \( V \) if \( f \) is \( C^p \) on \( V \) for all \( p \in \mathbb{N} \) (\( f \) is smooth).

Chain rule 1: Suppose \( f : (a, b) \to \mathbb{R}^n \), \( g : \mathbb{R}^n \to \mathbb{R} \), then

\[
\frac{d}{dt}(g \circ f)_{t_0} = D(G)_{f(t_0)}D(f)_{t_0} = (b_1, \ldots, b_n) \begin{pmatrix} f'_1(t_0) \\ f'_2(t_0) \\ \vdots \\ f'_n(t_0) \end{pmatrix}
\]

\[
= \sum_{i=1}^n \left( \frac{\partial g}{\partial x_i} \right)_{f(t_0)} \left( \frac{df}{dt} \right)_{t_0}
\]

Ex: \( f(t) = (t^2, \sin(t)) \), \( D(f) = \begin{pmatrix} 2t \\ \cos(t) \end{pmatrix} \)

\( g(x, y) = x + y^3 \), \( D(g) = (1, 3y^2) \)

\( (g \circ f)(t) = g(t^2, \sin(t)) = t^2 + \sin^3(t) \)

\( (g \circ f)'(t) = 2t_0 + 3\sin^2(t_0)\cos(t_0) \)

\( D(g)_{f(t_0)}D(f)_{t_0} = (1, 3\sin^2(t_0)) \begin{pmatrix} 2t_0 \\ \cos(t_0) \end{pmatrix} \),
Defn: $U$ is starlike with respect to $a$ if $x \in U$ implies $ax \subset U$

Thm 1.5 (Mean Value Theorem) Let $g$ by a differentiable function on an open set $U \subset \mathbb{R}^n$. Let $a \in U$ and suppose $U$ is starlike with respect to $a$. Then given $x \in U$, there exists $c \in \mathbb{R}$, $0 < t_0 < 1$ such that

$$g(x) - g(a) = \sum_{i=1}^{n} \left( \frac{\partial g}{\partial x_i} \right) p(x_i - a_i)$$

where $p = a + t_0(x - a)$

Cor 1.6: If $|\frac{\partial g}{\partial x_i}| < K$ on $U$ for all $i$, then for all $x \in U$,

$$|g(x) - g(a)| < K \sqrt{n}||x - a||$$

Cor 1.7 If $f \in C^r$ on $U$, then $\frac{\partial^k g}{\partial x_{i_1}\partial x_{i_2}...\partial x_{i_k}} = \frac{\partial^k g}{\partial x_{j_1}\partial x_{j_2}...\partial x_{j_k}}$ where $(j_1, j_2, ..., j_k)$ is a permutation of $(i_1, i_2, ..., i_k)$

2.2: $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$

Let $\pi_i: \mathbb{R}^m \rightarrow \mathbb{R}, \pi_i(x) = x_i$

$f = (f_1, ..., f_m)$ where $f_i = \pi_i \circ f$

$f$ continuous iff $f_i$ continuous for all $i$

$f \in C^r$ iff $f_i \in C^r$ for all $i$

$f \in C^\infty$ iff $f_i \in C^\infty$ for all $i$

Defn: The Jacobian matrix of $f$ at $a$ is

$$\left[ \frac{\partial f_i}{\partial x_j} (a) \right]_{m \times n} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} (a) & ... & \frac{\partial f_1}{\partial x_n} (a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} (a) & ... & \frac{\partial f_m}{\partial x_n} (a) \end{bmatrix}$$

2.1 Let $V$ be an open subset of $\mathbb{R}^n$, $a \in V$, $f: V \rightarrow \mathbb{R}^m$. Then $f$ is differentiable at $a$ if and only if there is a matrix $T$ and a function $\epsilon : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $\lim_{h \rightarrow 0} \epsilon(h) = 0$ and

$$f(a + h) - f(a) = T(h) + \|h\| \epsilon(h)$$

Or equivalently, there exists an $m$-tuple, $R(x, a) = (r_1(x, a), r_2(x, a), ..., r_m(x, a))$ such that $\lim_{x \rightarrow a} \|R(x, a)\| = 0$ and

$$f(x) = f(a) + T(x - a) + \|x - a\| R(x, a)$$
Thm 2.2: Let $f$ be a differentiable function on an open set $U \subset \mathbb{R}^n$. Let $a \in U$ and suppose $U$ is starlike with respect to $a$. If $\left| \frac{\partial f_i}{\partial x_i} \right| < K$ on $U$ for all $i, j$, then for all $x \in U$,

$$||f(x) - f(a)|| < K \sqrt{nm} ||x - a||$$

Proof: $||f(x) - f(a)|| = \sqrt{\sum_{i=1}^{m} (f_i(x) - f_i(a))^2}$

$$< \sqrt{\sum_{i=1}^{m} (K \sqrt{n} ||x - a||)^2} = \sqrt{m (K \sqrt{n} ||x - a||)^2} = K \sqrt{nm} ||x - a||$$

Thm 2.3 (Chain rule): Suppose $U \subset R^m$ is open and $f : U \to V \subset \mathbb{R}^m$, $g : V \to \mathbb{R}^p$. Let $h = g \circ f$. Suppose $f$ is differentiable at $a \in U$ and $g$ is differentiable at $f(a) \in V$. Then $h$ is differentiable at $a \in U$ and $D(h)_a = D(G)_{f(a)} D(f)_a$.

Cor 2.4: If $f, g \in C^r$ on $U, V$ respectively, then $h = g \circ f \in C^r$. 

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