HW 2.1: 2, 8 (due Friday, next week)

$f : \mathbf{R}^n \to \mathbf{R}$ is differentiable at a if

$$lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - f(\mathbf{a}) - T(\mathbf{x}-\mathbf{a})}{||\mathbf{x}-\mathbf{a}||} = 0$$

$$lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - f(\mathbf{a}) - \Sigma b_i(x_i - a_i)}{||\mathbf{x}-\mathbf{a}||} = 0$$

$y = f(\mathbf{a}) + \Sigma b_i(x_i - a_i)$ approximates $y = f(\mathbf{x})$

$f(\mathbf{x}) = f(\mathbf{a}) + T(\mathbf{x}-\mathbf{a}) + ||\mathbf{x}-\mathbf{a}||r(\mathbf{x},\mathbf{a})$
where $lim_{\mathbf{x}\to\mathbf{a}} r(\mathbf{x},\mathbf{a}) = 0$

Thm 1.1: If f is differentiable at a, then

1.) f is continuous at a.

2.) All partial derivatives exist at a.

3.) $b_i = (\frac{\partial f}{\partial x_i})_a$

Proof: 1.) $lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{a}) + T(\mathbf{x}-\mathbf{a}) + ||\mathbf{x}-\mathbf{a}||r(\mathbf{x},\mathbf{a}) =$

2,3.) $\frac{\partial f}{\partial x_j}(\mathbf{a}) = lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{e_j}) - f(\mathbf{a})}{h}$

$= lim_{h\to 0} \frac{f(\mathbf{a}) + T(\mathbf{a}+h\mathbf{e_j}-\mathbf{a}) + ||\mathbf{a}+h\mathbf{e_j}-\mathbf{a}||r(\mathbf{a}+h\mathbf{e_j},\mathbf{a}) - f(\mathbf{a})}{h}$

$= lim_{h\to 0} \frac{T(h\mathbf{e_j}) + |h|r(\mathbf{a}+h\mathbf{e_j},\mathbf{a})}{h} = lim_{h\to 0} \frac{hT(\mathbf{e_j}) + |h|r(\mathbf{a}+h\mathbf{e_j},\mathbf{a})}{h}$

Thm 1.3: If $\frac{\partial f}{\partial x_j}$ exist for all j in a nbhd of a and if they are continuous at a, then f is differentiable at a.

Defn: Let V be a nonempty open subset of R^n, $f : V \to R^m$, $p \in \mathbf{N}$.

i.) f is C^p on V is each partial derivative of order $k \leq p$ exists and is continuous on V.

ii.) f is C^∞ on V if f is C^p on V for all $p \in \mathbf{N}$ (f is *smooth*).

Chain rule 1: Suppose $f : (a,b) \to \mathbf{R}^n$, $g : \mathbf{R}^n \to \mathbf{R}$, then

$$\frac{d}{dt}(g \circ f)_{t_0} = D(G)_{f(t_0)} D(f)_{t_0} = (b_1, ..., b_n) \begin{pmatrix} f_1'(t_0) \\ f_2'(t_0) \\ ... \\ f_n'(t_0) \end{pmatrix}$$

$$= \Sigma_{i=1}^n (\frac{\partial g}{\partial x_i})_{f(t_0)} (\frac{df_i}{dt})_{t_0}$$

Ex: $f(t) = (t^2, sin(t))$, $D(f) = \begin{pmatrix} 2t \\ cos(t) \end{pmatrix}$

$g(x,y) = x + y^3$, $D(g) = (1, 3y^2)$

$(g \circ f)(t) = g(t^2, sin(t)) = t^2 + sin^3(t)$

$(g \circ f)'(t) = 2t_0 + 3sin^2(t_0)cos(t_0)$

$D(g)_{f(t_0)} D(f)_{t_0} = (1, 3sin^2(t_0)) \begin{pmatrix} 2t_0 \\ cos(t_0) \end{pmatrix},$

Defn: U is starlike with respect to $\mathbf{a}$ if $\mathbf{x} \in U$ implies $\overline{\mathbf{ax}} \subset U$

Thm 1.5 (Mean Value Theorem) Let g by a differentiable function on an open set $U \subset \mathbf{R}^n$. Let $\mathbf{a} \in U$ and suppose U is starlike with respect to $\mathbf{a}$. Then given $\mathbf{x} \in U$, there exists $c \in \mathbf{R}$, $0 < t_0 < 1$ such that

$g(\mathbf{x}) - g(\mathbf{a}) = \Sigma_{i=1}^n (\frac{\partial g}{\partial x_i})_{\mathbf{p}}(x_i - a_i)$

where $\mathbf{p} = \mathbf{a} + t_0(\mathbf{x} - \mathbf{a})$

Cor 1.6: If $|\frac{\partial g}{\partial x_i}| < K$ on U for all i, then for all $\mathbf{x} \in U$,

$$|g(\mathbf{x}) - g(\mathbf{a})| < K\sqrt{n}||\mathbf{x} - \mathbf{a}||$$

Cor 1.7 If $f \in C^r$ on U, then $\frac{\partial^k g}{\partial x_{i_1}\partial x_{i_2}...\partial x_{i_k}} = \frac{\partial^k g}{\partial x_{j_1}\partial x_{j_2}...\partial x_{j_k}}$ where $(j_1, j_2, ..., j_k)$ is a permutation of $(i_1, i_2, ..., i_k)$

2.2: $f : \mathbf{R}^n \to \mathbf{R}^m$

Let $\pi_i : \mathbf{R}^m \to \mathbf{R}, \pi_i(\mathbf{x}) = x_i$

$f = (f_1, ..., f_m)$ where $f_i = \pi_i \circ f$

f continuous iff f_i continuous for all i

$f \in C^r$ iff $f_i \in C^r$ for all i

$f \in C^\infty$ iff $f_i \in C^\infty$ for all i

Defn: The **Jacobian matrix of f at $\mathbf{a}$** is

$$\left[\frac{\partial f_i}{\partial x_j}(\mathbf{a})\right]_{m\times n} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & ... & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ . & . & . \\ . & . & . \\ . & . & . \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & ... & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}$$

2.1 Let V be an open subset of R^n, $\mathbf{a} \in V$, $f : V \to R^m$. Then f is differentiable at $\mathbf{a}$ if and only if there is a matrix T and a function $\epsilon : R^n \to R^m$ such that $lim_{\mathbf{h}\to 0}\epsilon(\mathbf{h}) = \mathbf{0}$ and

$$f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) = T(\mathbf{h}) + ||\mathbf{h}||\epsilon(\mathbf{h})$$

Or equivalently, there exists an m-tuple,
$R(\mathbf{x}, \mathbf{a}) = (r_1(\mathbf{x}, \mathbf{a}), r_2(\mathbf{x}, \mathbf{a}), ..., r_m(\mathbf{x}, \mathbf{a})$
such that $lim_{\mathbf{x}\to a}||R(\mathbf{x}, \mathbf{a})|| = 0$ and

$$f(\mathbf{x}) = f(\mathbf{a}) + T(\mathbf{x} - \mathbf{a}) + ||\mathbf{x} - \mathbf{a}||R(\mathbf{x}, \mathbf{a})$$

Thm 2.2: Let f by a differentiable function on an open set $U \subset \mathbf{R}^n$. Let $\mathbf{a} \in U$ and suppose U is starlike with respect to $\mathbf{a}$. If $|\frac{\partial f_i}{\partial x_i}| < K$ on U for all i, j, then for all $\mathbf{x} \in U$,

$$||f(\mathbf{x}) - f(\mathbf{a})|| < K\sqrt{nm}||\mathbf{x} - \mathbf{a}||$$

Proof: $||f(\mathbf{x}) - f(\mathbf{a})|| = \sqrt{\Sigma_{i=1}^m (f_i(\mathbf{x}) - f_i(\mathbf{a}))^2}$

$$< \sqrt{\Sigma_{i=1}^m (K\sqrt{n}||\mathbf{x} - \mathbf{a}||)^2} = \sqrt{m(K\sqrt{n}||\mathbf{x} - \mathbf{a}||)^2}$$
$$= K\sqrt{nm}||\mathbf{x} - \mathbf{a}||$$

Thm 2.3 (Chain rule): Suppose $U \subset R^m$ is open and $f : U \to V \subset \mathbf{R}^m$, $g : V \to \mathbf{R}^p$. Let $h = g \circ f$. Suppose f is differentiable at $a \in U$ and g is differentiable at $f(a) \in V$. Then h is differentiable at $a \in U$ and $D(h)_a = D(G)_{f(a)} D(f)_a$.

Cor 2.4: If $f, g \in C^r$ on U, V respectively, then $h = g \circ f \in \mathbf{C}^r$.