Urysohn Lemma: If X is normal then for any A, B disjoint closed sets in X, there exists a continuous function \( f : X \to [0, 1] \) such that \( f(A) = \{0\} \) and \( f(B) = \{1\} \)

Proof: Suppose A, B disjoint closed sets in X.

Note by listing Q, we can do induction on the rationals, defining a U_q for each rational number (or in our case \( Q \cap [0, 1] \)).

Choose a bijective function \( g : N \to Q \cap [0, 1] \) such that \( g(1) = 1 \) and \( g(2) = 0 \). Let \( g(n) = q_n \)

Define \( U_i = X \) for all \( i > 1 \).

\( q_1 = 1 \): Define \( U_1 = X - B \)

\( q_2 = 0 \): Since X is normal, there exists an open set \( U_0 \) such that \( A \subset U_0 \subset \overline{U_0} \subset U_1 \).

Similarly there exists an open set \( U_{q_3} \) such that \( \overline{U_0} \subset U_{q_3} \subset \overline{U_{q_3}} \subset U_1 \).

Suppose \( U_{q_1}, ..., U_{q_{n-1}} \) have been defined such that \( U_{q_i} \) is open and if \( q_i < q_j \) then \( \overline{U_{q_i}} \subset U_{q_j} \)

Define \( U_{q_n} \):

Let \( s = \max \{q_i \mid q_i < q_n, i = 1, ..., n-1\} \)

Let \( t = \min \{q_i \mid q_i > q_n, i = 1, ..., n-1\} \)

Since X is normal, there exists an open set \( U_{q_n} \) s. t. \( \overline{U_s} \subset U_{q_n} \subset \overline{U_{q_n}} \subset U_t \).

Thus if \( q_n < q_i \), then \( q_n < t \leq q_i \).

Thus if \( q_i < q_n \), then \( q_i \leq s < q_n \).

Hence we have defined \( U_p \) for all \( p \in Q \cup [0, \infty) \) such that \( U_p \) is open for all \( p \) and if \( p < q \) then \( \overline{U_p} \subset U_q \).

Define \( f : X \to \mathcal{R} \) by

\[ f(x) = \inf \{ p \mid x \in U_p, \ p \in Q \cap [0, \infty) \} \]

Note infimum exists since \( x \in X = U_{1,1} \) implies \( 1.1 \in \{ p \mid x \in U_p, \ p \in Q \cap [0, \infty) \} \) and \( \{ p \mid x \in U_p, \ p \in Q \cap [0, \infty) \} \) is bounded below by 0.

If \( x \in A \), then \( x \in U_0 \). Thus \( f(x) = \inf \{ p \mid x \in U_p, \ p \in Q \cap [0, \infty) \} = 0 \)

If \( x \in B \), then \( x \in U_p = X \forall p > 1 \). But \( x \notin U_1 = X - B \).

Since \( \overline{U_q} \subset U_1 \) for all \( q < 1 \), \( x \notin U_q \) for all \( q \leq 1 \).

Thus \( f(x) = \inf \{ p \mid x \in U_p, \ p \in Q \cap [0, \infty) \} = 1 \)

Since \( x \in U_p = X \) for all \( p > 1 \).

\( f(x) = \inf \{ p \mid x \in U_p, \ p \in Q \cap [0, \infty) \} \leq 1 \forall x \in X \).

Since \( \inf \{ p \mid p \in Q \cap [0, \infty) \} = 0 \),

\( \inf \{ p \mid x \in U_p, \ p \in Q \cap [0, \infty) \} \geq 0 \) for all \( x \in X \).

Thus \( f(X) \subset [0, 1] \) and hence \( f : X \to [0, 1] \).

Claim: \( f : X \to [0, 1] \) is continuous.
\[ f : X \to [0, 1]. \] is continuous if and only if \( f : X \to \mathbb{R} \) is continuous.

Claim: \( f : X \to \mathbb{R} \) is continuous.

Take \((a, b) \subset \mathbb{R}\) and \(x \in f^{-1}(a, b)\). Then \( f(x) \in (a, b) \).

Take \(p, q \in \mathbb{Q}\) such that \( a < p < f(x) < q < b \).

Claim: \( x \in U_q - \overline{U_p} \subset f^{-1}(a, b) \).

subclaim 1: \( z \in \overline{U_r} \) implies \( f(z) \leq r \)

Suppose \( z \in \overline{U_r} \). If \( s > r \), then \( z \in \overline{U_r} \subset U_s \).

Hence \( f(z) = \inf \{p \mid z \in U_p, \, p \in \mathbb{Q} \cap [0, \infty)\} \leq \inf \{s \in \mathbb{Q} \mid s > r\} = r \).

subclaim 2: \( z \notin U_r \) implies \( f(z) \geq r \)

Suppose \( z \notin U_r \). If \( s < r \), then \( \overline{U_s} \subset U_r \).

Then \( z \notin U_r \) implies \( z \notin \overline{U_s} \).

Thus \( r \) is a lower bound for \( \{p \mid z \in U_p, \, p \in \mathbb{Q} \cap [0, \infty)\} \).

Hence \( f(z) \geq r \).

Thus
\[(f(z) > r \text{ implies } z \notin \overline{U_r}) \& (f(z) < r \text{ implies } z \in U_r).\]

Hence \( p < f(x) < q \) implies \( x \in U_q - \overline{U_p} \).

If \( z \in U_q - \overline{U_p} \), then \( z \in \overline{U_q} \) and hence \( f(z) \leq q < b \). Also, \( z \notin \overline{U_p} \) implies \( z \notin U_p \), and hence \( f(z) \geq p > a \). Thus \( f(z) \in (a, b) \) and \( U_q - \overline{U_p} \subset f^{-1}(a, b) \). \( U_q - \overline{U_p} \) is open.

Hence \( f \) is continuous.

Defn: If \( f : X \to [0, 1] \) is a fn s. t. \( f(A) = \{0\} \& f(B) = \{1\} \) for \( A, B \subset X \), then \( f \) is said to separate \( A \& B \).

Suppose \( X \) is \( T_1 \). Then \( X \) is \( T_4 \) iff for each pair of disjoint closed subsets of \( X \), there exists a continuous function \( f : X \to [0, 1] \) which separates them.

Defn: \( X \) is completely regular (or \( T_{3.5} \)) if \( X \) is \( T_1 \) and for each \( x \in X \) and for each closed set \( A \subset X \) such that \( x \notin A \), there exists a continuous function \( f : X \to [0, 1] \) such that \( f(A) = \{0\} \) and \( f(x) = 1 \).

36: Imbeddings of Manifolds

\( X \) is locally Euclidean if for all \( x \in X \), there exists \( U \) open such that \( x \in U \), and there exists a homeomorphism \( f : U \to f(U) \subset \mathbb{R}^m \) where \( f(U) \) is open in \( \mathbb{R}^m \).

Ex: \((0, 1)\) is locally Euclidean, but \([0, 1]\) is NOT locally Euclidean.

\( X \) is an \( m-\)manifold if
(1) \( X \) is locally Euclidean
(2) \( X \) is \( T_2 \)
(3) \( X \) 2nd countable.

A 1-manifold is a curve (ex: the circle \( S^1 \))

A 2-manifold is a surface

Orientable surfaces: sphere \( S^2 \), torus \( T^2 \), connected sum of tori \( T^2 \# \ldots \# T^2 \).

Non-orientable surfaces: projective plane \( \mathbb{R}P^2 \), Klein bottle, \( \mathbb{R}P^2 \# \mathbb{R}P^2 \), connected sum of projective planes \( \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2 \).