Urysohn Lemma: If X is normal then for any A, B disjoint closed sets in X, there exists a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$

Proof: Suppose A, B disjoint closed sets in X,

Note by listing $\mathcal{Q}$, we can do induction on the rationals, defining a U_q for each rational number (or in our case $\mathcal{Q} \cap [0,1]$

Choose a bijective function $g : \mathcal{N} \to \mathcal{Q} \cap [0,1]$ such that $g(1) = 1$ and $g(2) = 0$. Let $g(n) = q_n$

Define $U_i = X$ for all $i > 1$.
$q_1 = 1$: Define $U_1 = X - B$

$q_2 = 0$: Since X is normal, there exists an open set U_0 such that $A \subset U_0 \subset \overline{U_0} \subset U_1$.

Similarly there exists an open set U_{q_3} *such that*

$$\overline{U_0} \subset U_{q_3} \subset \overline{U_{q_3}} \subset U_1.$$

Suppose $U_{q_1}, ..., U_{q_{n-1}}$ have been defined such that U_{q_i} is open and if $q_i < q_j$ then $\overline{U_{q_i}} \subset U_{q_j}$

Define U_{q_n}:

$$\text{Let } s = max\{q_i \mid q_i < q_n, i = 1, ..., n-1\}$$

$$\text{Let } t = min\{q_i \mid q_i > q_n, i = 1, ..., n-1\}$$

Since X is normal, there exists an open set U_{q_n} s. t.

$$\overline{U_s} \subset U_{q_n} \subset \overline{U_{q_n}} \subset U_t.$$

Thus if $q_n < q_i$, then $q_n < t \leq q_i$.

$$\text{Thus } \overline{U_{q_n}} \subset U_t \subset \overline{U_t} \subset U_{q_i}$$

Thus if $q_i < q_n$, then $q_i \leq s < q_n$.

$$\text{Thus } \overline{U_{q_i}} \subset U_s \subset \overline{U_s} \subset U_{q_n}$$

Hence we have defined U_p for all $p \in \mathcal{Q} \cup [0, \infty)$ such that U_p is open for all p and if $p < q$ then $\overline{U_p} \subset U_q$.

Define $f : X \to \mathcal{R}$ by

$$f(x) = inf\{p \mid x \in U_p,\ p \in \mathcal{Q} \cap [0, \infty)\}$$

Note infimum exists since
$x \in X = U_{1.1}$ implies $1.1 \in \{p \mid x \in U_p,\ p \in \mathcal{Q} \cap [0, \infty)\}$ and $\{p \mid x \in U_p,\ p \in \mathcal{Q} \cap [0, \infty)\}$ is bounded below by 0.

If $x \in A$, then $x \in U_0$.

$$\text{Thus } f(x) = inf\{p \mid x \in U_p,\ p \in \mathcal{Q} \cap [0, \infty)\} = 0$$

If $x \in B$, then $x \in U_p = X \ \forall\, p > 1$. But $x \notin U_1 = X - B$. Since $\overline{U_q} \subset U_1$ for all $q < 1$, $x \notin U_q$ for all $q \leq 1$.

$$\text{Thus } f(x) = inf\{p \mid x \in U_p,\ p \in \mathcal{Q} \cap [0, \infty)\} = 1$$

Since $x \in U_p = X$ for all $p > 1$.

$$f(x) = inf\{p \mid x \in U_p,\ p \in \mathcal{Q} \cap [0, \infty)\} \leq 1 \ \forall x \in X.$$

Since $inf\{p \mid p \in \mathcal{Q} \cap [0, \infty)\} = 0$,

$$inf\{p \mid x \in U_p,\ p \in \mathcal{Q} \cap [0, \infty)\} \geq 0 \text{ for all } x \in X.$$

Thus $f(X) \subset [0,1]$ and hence $f : X \to [0,1]$.

Claim: $f : X \to [0,1]$ is continuous.

$f : X \to [0,1]$. is continuous if and only if $f : X \to \mathcal{R}$ is continuous.

Claim: $f : X \to \mathcal{R}$ is continuous.

Take $(a,b) \subset \mathcal{R}$ and $x \in f^{-1}(a,b)$. Then $f(x) \in (a,b)$.

Take $p, q \in \mathcal{Q}$ such that $a < p < f(x) < q < b$.

Claim: $x \in U_q - \overline{U_p} \subset f^{-1}(a,b)$.

subclaim 1: $z \in \overline{U_r}$ implies $f(z) \leq r$
Suppose $z \in \overline{U_r}$. If $s > r$, then $z \in \overline{U_r} \subset U_s$
Hence $f(z) = inf\{p \mid z \in U_p,\ p \in \mathcal{Q} \cap [0,\infty)\}$
$\leq inf\{s \in \mathcal{Q} \mid s > r\} = r$

subclaim 2: $z \notin U_r$ implies $f(z) \geq r$.
Suppose $z \notin U_r$. If $s < r$, then $\overline{U_s} \subset U_r$.
Then $z \notin U_r$ implies $z \notin U_s$.
Thus r is a lower bound for $\{p \mid z \in U_p,\ p \in \mathcal{Q} \cap [0,\infty)\}$.
Hence $f(z) \geq r$.

Thus
$(f(z) > r$ implies $z \notin \overline{U_r})$ & $(f(z) < r$ implies $z \in U_r)$.

Hence $p < f(x) < q$ implies $x \in U_q - \overline{U_p}$

If $z \in U_q - \overline{U_p}$, then $z \in \overline{U_q}$ and hence $f(z) \leq q < b$. Also, $z \notin \overline{U_p}$ implies $z \notin U_p$, and hence $f(z) \geq p > a$. Thus $f(z) \in (a,b)$ and $U_q - \overline{U_p} \subset f^{-1}(a,b)$. $U_q - \overline{U_p}$ is open. Hence f is continuous.

Defn: If $f : X \to [0,1]$ is a fn s. t. $f(A) = \{0\}$&$f(B) = \{1\}$ for $A, B \subset X$, then f is said to *separate* A&B.

Suppose X is T_1. Then X is T_4 iff for each pair of disjoint closed subsets of X, there exists a continuous function $f : X \to [0,1]$ which separates them.

Defn: X is *completely regular* (or $T_{3.5}$) if X is T_1 and for each $x \in X$ and for each closed set A in X such that $x \notin A$, there exists a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(x_0) = 1$

36: Imbeddings of Manifolds

X is *locally Euclidean* if for all $x \in X$, there exists U open such that $x \in U$, and there exists a homeomorphism $f : U \to f(U) \subset \mathbf{R}^m$ where $f(U)$ is open in $\mathbf{R}^m$.

Ex: $(0,1)$ is locally Euclidean, but $[0,1]$ is NOT locally Euclidean.

X is an $m-$*manifold* if
(1) X is locally Euclidean
(2) X is T_2
(3) X 2nd countable.

A 1-manifold is a *curve* (ex: the circle S^1)

A 2-manifold is a *surface*

Orientable surfaces: sphere S^2, torus T^2, connected sum of tori $T^2\#...\#T^2$, Non-orientable surfaces: projective plane $\mathbf{R}P^2$, Klein bottle, $\mathbf{R}P^2\#\mathbf{R}P^2$, connected sum of projective planes $\mathbf{R}P^2\#...\#\mathbf{R}P^2$.