

Definition: A **subbasis** \mathcal{S} for a topology on X is a collection of subsets of X whose union equals X .

Lemma: If \mathcal{S} is a subbasis for a topology on X , then $\mathcal{B} = \{\cap_{i=1}^n S_i \mid S_i \in \mathcal{S}\}$ is a basis for a topology.

Note that \mathcal{B} is a collection of subsets of X ($S_i \subset X$ implies $\cap_{i=1}^n S_i \subset X$).

(1) Show for each $x \in X$, there is at least one basis element B containing x .

Proof: Take $x \in X$. Find $B \in \mathcal{B}$ such that $x \in B$.

We will first show that $\mathcal{S} \subset \mathcal{B}$: Suppose that $S_1 \in \mathcal{S}$. Then $S_1 = \cap_{n=1}^1 S_1$. Thus S_1 is a finite intersection of elements of \mathcal{S} since 1 is a finite number. Hence $S_1 \in \mathcal{B}$.

Since \mathcal{S} is a collection of subsets of X whose union equals X , there exists an $S \in \mathcal{S} \subset \mathcal{B}$ such that $x \in S$. Hence, there is at least one basis element B containing x .

(2) Show that if $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that

$$x \in B_3 \subset B_1 \cap B_2.$$

The topology \mathcal{T} generated by a basis \mathcal{B} is defined as follows: U is open if and only if for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

Negation:

U is NOT open in \mathcal{T} iff the following is false: for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

U is NOT open in \mathcal{T} iff there exists an $x \in U$, such that the following is false: there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

U is NOT open in \mathcal{T} iff there exists an $x \in U$, such that for all $B \in \mathcal{B}$ the following is false: $x \in B \subset U$.

U is NOT open in \mathcal{T} iff there exists an $x \in U$, such that for all $B \in \mathcal{B}$ the following is false: $x \in B$ and $B \subset U$.

U is NOT open in \mathcal{T} iff there exists an $x \in U$, such that for all $B \in \mathcal{B}$ either $x \notin B$ OR $B \not\subset U$.

U is NOT open in \mathcal{T} iff there exists an $x \in U$, such that for all $B \in \mathcal{B}$, $x \in B$ implies $B \not\subset U$.