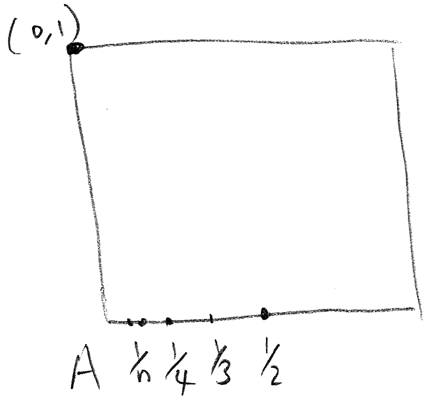
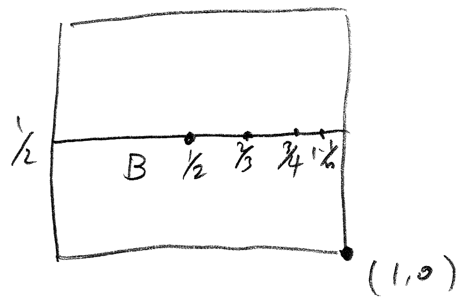
17.18.



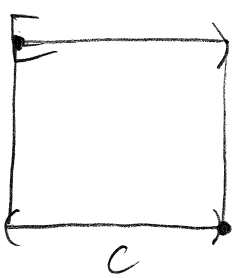
$\bar{A} = A \cup (0,1)$

since any open neighborhood of $(0,1)$ is an interval with right end point of form (a,b) and $a>0$, so $\exists\, n$ s.t. $(\frac{1}{n},0) < (a,b)$.

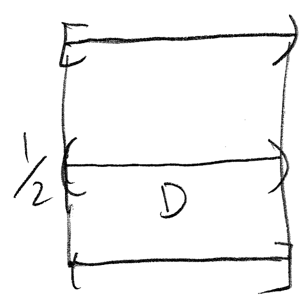


$\bar{B} = B \cup (1,0)$.

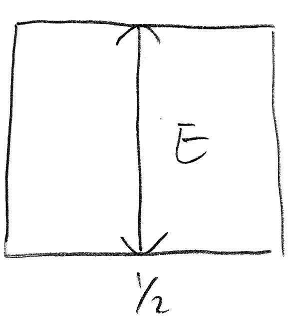
Same reason.



$\bar{C} = \{x \times 1 \mid 0 \le x < 1\} \cup C \cup \{(1,0)\}$



$\bar{D} = D \cup \{x \times 1 \mid 0 \le x < 1\} \cup \{x \times 0 \mid 0 < x \le 1\}$



$\bar{E} = \{\frac{1}{2} \times y \mid 0 \le y \le \frac{1}{2}\}$

17.19.

a) i) $\text{Int} A \cap Bd(A) = \emptyset$.

suppose not, $x \in \text{Int} A \cap Bd(A)$

$\Rightarrow$ $x \in \text{Int} A$ and $x \in \bar{A} \cap \overline{(X-A)}$.

But $x \in \text{Int} A \Rightarrow \exists$ an open neighborhood U s.t $x \in U \subset A$

$\Rightarrow U \cap X - A = \emptyset \Rightarrow x \notin \overline{X-A}$ $\rightarrow\leftarrow$.

ii) $\bar{A} = \text{Int} A \cup Bd(A)$

Take $x \in \bar{A} \setminus Bd(A)$, since $Bd(A) = \bar{A} \cap \overline{(X-A)}$.

$\Rightarrow x \notin \overline{X-A}$. $\Rightarrow \exists$ an open neighborhood U of x st

$U \cap (X-A) = \emptyset \Rightarrow U \subset A \Rightarrow x \in \text{Int} A$

b) $Bd A = \emptyset \Leftrightarrow$ A is both open and closed.

$\Rightarrow$ if $Bd A = \emptyset$. by a) $\bar{A} = \text{Int}(A)$. but trivially

$\text{Int}(A) \subset A \subset \bar{A}$ $\Rightarrow$ $A = \text{Int}(A) = \bar{A}$. $\Rightarrow$ A is both open & closed.

since $\text{Int}(A)$ is open, and $\bar{A}$ is closed

$\Leftarrow$ if A is both open and closed, we have

$A = \text{Int}(A)$ & $A = \bar{A}$ $\Rightarrow$ $Bd(A) = \emptyset$.

c) U is open $\Leftrightarrow$ $Bd U = \bar{U} - U$

'$\Rightarrow$' U is open $\Rightarrow U = \text{Int} U$. , $\bar{U} = \text{Int} U \cup Bd U$ $\Rightarrow$ $Bd U = \bar{U} - U$.

since the union is disjoint.

'$\Leftarrow$' if $Bd U = \bar{U} - U$, then $U = \text{Int} U \Rightarrow U$ is open

d) Not true for example:

$U = (0,1) \cup (1,2)$ $\bar{U} = [0, 2]$

$\text{Int}(\bar{U}) = (0,2)$. and $\text{Int}(U) = U$

18.9

a) Pasting lemma & Induction

- If there are two closed sets, pasting lemma $\Rightarrow$ f is continuous
- Assume it is true for $k-1$ closed sets.

Then for k closed sets $A_{\alpha_1}, \ldots A_{\alpha_k}$.

we can write it as

$\tilde{A}_1 = A_{\alpha_1} \cup \cdots \cup A_{\alpha_{k-1}}$ $\leftarrow$ closed $\quad f|_{\tilde{A}_1}$ is continuous By assumption.

$\tilde{A}_2 = A_{\alpha_k}$ $\leftarrow$ closed $\quad f|_{\tilde{A}_2}$ is continuous by hypothesis.

by pasting lemma $\Rightarrow$ f is continuous.

b) $f: (\mathbb{R}, \text{finite complement topology}) \to \mathbb{R}_s$

$f(x) = x \qquad A_\alpha = \{x\}$ single point set

A_α's are <u>closed</u>. $f|_{A_\alpha}$ is <u>continuous</u> (since constant)

but f is not continuous.

$f^{-1}((0,1)) = (0,1)$ $\quad (0,1)$ is not open in finite complement topology.

c) We will prove f is continuous at every $x \in X$.

Indeed, $\exists\, \mathcal{U}$, an open neighborhood of x. s.t only finite many A_α's intersect $\mathcal{U}$, say they are $A_{\alpha_1}, \ldots, A_{\alpha_n}$.

claim: $\mathcal{U} \subset \bigcup_{i=1}^{n} A_{\alpha_i}$, since $\cup A_\alpha = X$, but other A_α's do not intersect $\mathcal{U}$.

Then $B_{\alpha_i} = \mathcal{U} \cap A_{\alpha_i}$ $\leftarrow$ closed in $\mathcal{U}$.

$f|_{B_{\alpha_i}}$ is continuous. $\quad$ apply a) $\Rightarrow$ f is continuous on $\mathcal{U}$.

18.13 Pf: Suppose there are two different extensions: g_1, g_2.

So $\exists\ x \in A'$, s.t $g_1(x) \neq g_2(x)$.

since Y is Hausdorff, $\exists$ disjoint neighborhood V_1, V_2 in Y: $g_1(x) \in V_1$, $g_2(x) \in V_2$, $V_1 \cap V_2 = \phi$.

$U_1 = g_1^{-1}(V_1)$, $U_2 = g_2^{-1}(V_2)$ are two neighborhoods of x in X.

So $U = U_1 \cap U_2$ is an open neighborhood of x.

since $x \in A' \Rightarrow \exists\ y \in A$ s.t $y \in U$

however we must have,

$$g_1(u) \subset g_1(U_1) = V_1, \quad g_2(u) \subset g_2(U_2) = V_2$$

and $V_1 \cap V_2 = \phi$. but $g_1(y) = g_2(y) = f(y)$, since $y \in A$.

$\Rightarrow$ a contradiction.

Supplementary Exercises:

1. '$\Rightarrow$' if H is topological

$$H \times H \to H \times H \to H$$
$$(x, y) \xrightarrow{\text{continuous}} (x, y^{-1}) \xrightarrow{\text{continuous}} xy^{-1}$$

thus $(x,y) \to xy^{-1}$ continuous.

'$\Leftarrow$' if $(x,y) \to xy^{-1}$ is continuous. then let $y = 1$

$\Rightarrow$ $x \to x^{-1}$ is continuous.

$$H \times H \to H \times H \to H$$
$$(x, y) \xrightarrow[(y \to y^{-1})]{\text{continuous}} (x, y^{-1}) \xrightarrow{\text{contiuous}} x(y^{-1})^{-1} = xy,$$

thus $(x,y) \to xy$ is continuous.

2. All operations are algebraic operations $(+, -, \times, -)$ so No problem with continuity.