Ph.D. Qualifying Exam in Topology

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Instructions. Do eight problems, four from each part. This is a closed book examination, you should have no books or paper of your own. Please do your work on the paper provided. Clearly number your pages corresponding to the problem you are working. When you start a new problem, start a new page; only write on one side of the paper. Make a cover page and indicate clearly which eight problems you want graded.

Always justify your answers unless explicitly instructed otherwise. You may use theorems if the problem is not a step in proving that theorem, but you need to state any theorems you use carefully.

Part A

1. Suppose that $Y \subset X$ is given the subspace topology from the topological space X. Prove that the nonempty sets C, D form a separation of Y if and only if $C \cup D = Y$, $C \cap D = \emptyset$, $C' \cap D = \emptyset$, and $C \cap D' = \emptyset$ where C' and D' are the limit points of C and D in X respectively.

2. Prove that if X is a regular second countable topological space then X is normal.

3. Suppose that $R \subset X \times X$ is an equivalence relation on the topological space X and $q : X \to \overline{X}$ is the corresponding quotient map onto the quotient space $\overline{X}$ with the quotient topology. Prove that if $q : X \to \overline{X}$ is open, then $\overline{X}$ is Hausdorff if and only if R is closed in $X \times X$.

4. Prove that if (X, d) is a metric space then X is normal.

5. Prove that if $f : X \to Y$ is a continuous, open map, X is compact, and Y is connected and Hausdorff then f is onto.

6. Give an example of a topological space X a subset A and a limit point m of A so that there is no sequence $a_n \in A$ with

$$\lim_{n \to \infty} a_n = m.$$

Part B

1. Let M be a smooth n-manifold and let $p \in M$. Recall, T_pM is the vector space of linear maps $L : C^\infty(M) \to \mathbb{R}$ so that for all $f, g \in C^\infty(M)$,

$$L(fg) = g(p)L(f) + f(p)Lg).$$

Prove that tangent vectors are local. That is prove that if $L \in T_pM$ and $f, g \in C^\infty(M)$ and there is an open neighborhood U of p so that $f|_U = g|_U$ then $L(f) = L(g)$. *You may use the existence of smooth partitions of unity subordinate to an open cover.*

2. Let $O(n)$ denote the set of $n \times n$ matrices A with real entries so that $AA^t = Id$. Prove that $O(n)$ is a smooth manifold and compute its tangent space at Id.

3. Prove the *local immersion theorem.* That is if $F : M \to N$ is a smooth map of smooth manifolds and F is an immersion at p then there exist coordinate charts (U, ϕ) for M, and (V, ψ) for N so that

   - $p \in U$
   - $F(U) \subset V$, and
   - $\psi \circ F \circ \phi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0)$ where there are $n - m$ zeroes.

4. Let T be the subset of $\mathbb{R}^3$ obtained by rotating a circle of radius 1 in the yz-plane centered at $(0, 2, 0)$ about the z-axis.

   a. Prove that T is a smooth submanifold of $\mathbb{R}^3$.

   b. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the map $\pi(x, y, z) = (x, y)$. Find the regular and singular values of $\pi|_T$.

5. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. Let $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$. Orient S^2 with the outwards normal. Compute

$$\int_{S^2} \omega.$$

6. Let M be a smooth n-manifold. Let $A^k(M)$ denote the smooth k-forms on M. Prove that if $L : A^*(M) \to A^*(M)$ is an R-linear map so that $L(A^k(M)) \subset A^{k+1}(M)$ and having the properties:

- If $f \in A^0(M)$ then $Lf = df$, that is Lf is the differential of f.
- If $\omega \in A^k(M)$ and $\eta \in A^l(M)$ then

  $$L(\omega \wedge \eta) = L(\omega) \wedge \eta + (-1)^k \omega \wedge L(\eta)$$

  and
- $L \circ L = 0$

then L is the exterior derivative. *You may assume that the exterior derivative satisfies the three properties above.*