Theorem:
The image of a compact space under a continuous map is compact.

Proof:

Let $f : X \to Y$ be a continuous function. We wish to show that:

$$X \text{ is compact } \Rightarrow f(X) \text{ is compact}$$

Let $\mathcal{A}$ be an open cover of $f(X)$ (where all the sets of $\mathcal{A}$ are open in $Y$).

Then the collection

$$\{f^{-1}(A) | A \in \mathcal{A}\}$$

is an open cover of $X$. Note that these sets are open because $f$ is continuous by assumption.

Since $X$ is compact, there exists a finite subcover:

$$\{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\}$$

which covers $X$. Therefore the collection

$$\{A_1, A_2, \ldots, A_n\}$$

forms a finite subcover of $\mathcal{A}$ which covers $f(X)$.

Therefore $f(X)$ is compact.

QED