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Lectures at Knots 96
edited by Shin'ichi Suzuki

pp. 73–93

KNOT TYPES OF SATELLITE KNOTS AND TWISTED KNOTS

KIMIHIKO MOTEGI

Let K be a knot inside a standardly embedded solid torus V in the 3-sphere S^3. In the following, for nontriviality, we assume that K cannot be lie in a 3-ball in V. Knotting the solid torus V in the shape of another knot as in Figure 0.1, as the image of K, we obtain a new knot K' in S^3. A knot obtained in such a manner is called a *satellite knot*. On the other hand, twisting the solid torus V several times, we get a new knot K'' in S^3. The purpose in this article is to give a survey of some aspects of the study of these constructions.

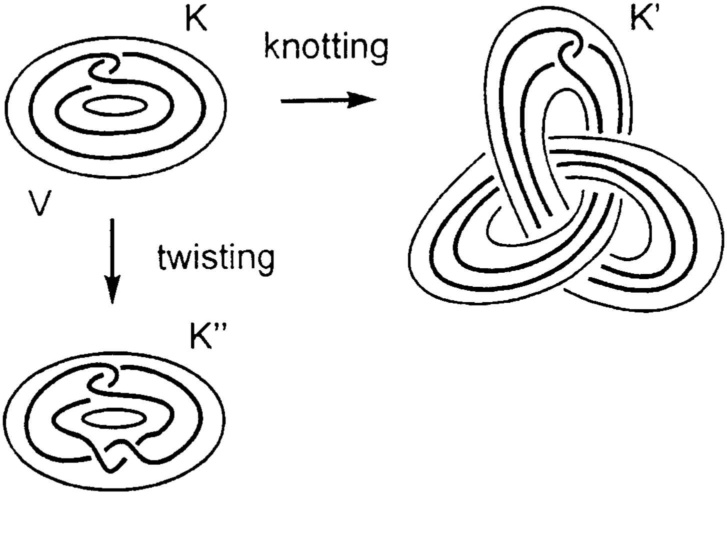

Figure 0.1

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1. Satellite knots obtained from a given pattern

Schubert [17] introduced the notion of the product of knots, and afterward generalized this to an operation "taking satellite" [18].

First we recall the construction of satellite knots. Let V be a standardly embedded solid torus in the oriented 3-sphere S^3 with the orientation induced from that of S^3, and let K be a knot in V, which cannot be contained in a 3-ball in V. Using an orientation preserving embedding $f : V \to S^3$ such that $f(V)$ is knotted in S^3, we can obtain a new knot $f(K)$ in S^3. We call the knot $f(K)$ a *satellite knot* and (V, K) a *pattern* (Figure 1.1).

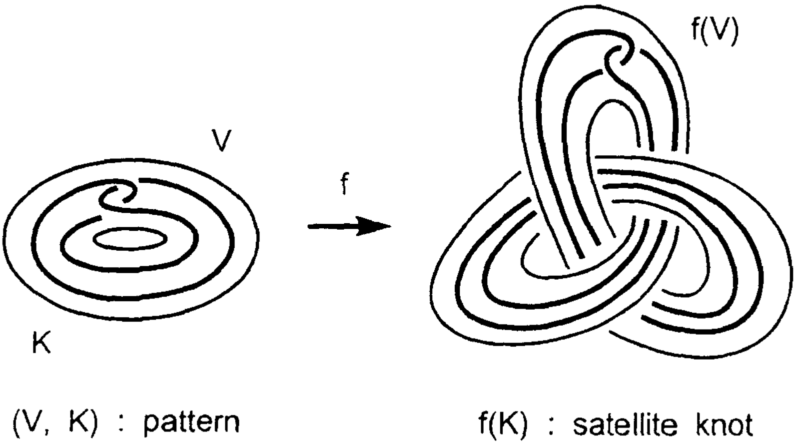


Figure 1.1

The *wrapping number* (resp. *winding number*) of K in V is defined to be the minimal geometric intersection number (resp. algebraic intersection number) of K and a meridian disk of V. We denote this number by $\mathrm{wrap}_V(K)$ (resp. $\mathrm{wind}_V(K)$).

For example, $\mathrm{wrap}_V(K) = 2$ and $\mathrm{wind}_V(K) = 0$ for the pattern (V, K) given by Figure 1.2.

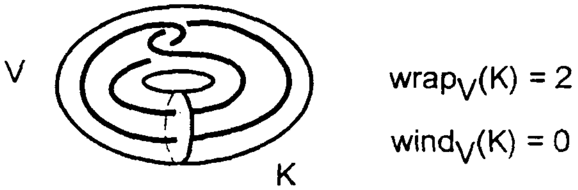


Figure 1.2

Throughout this section we assume that all knots are oriented and consider two knots K_1 and K_2 to be equivalent if and only if there is an orientation preserving homeomorphism $h : S^3 \to S^3$ which carries K_1 onto K_2 so that their orientations match. We write $K_1 \cong K_2$ if K_1 and K_2 are equivalent, and $-K$ denotes the knot obtained from K by inverting its orientation. For an orientation preserving embedding $f : V \to S^3$, we understand that $f(V)$ and $f(K)$ have orientations induced from that of V and K respectively via the embedding f.

The construction of a satellite knot depends on two parameters: the pattern (V, K) and the orientation preserving embedding $f : V \to S^3$. Let us choose a pattern (V, K). Then by changing the embedding as in Figure 1.3, we can obtain other satellite knots.

If two embeddings are isotopic, then clearly they define equivalent satellite knots. Conversely, can a satellite knot determine an isotopy class of embeddings of V into S^3? Precisely we consider the following problem.

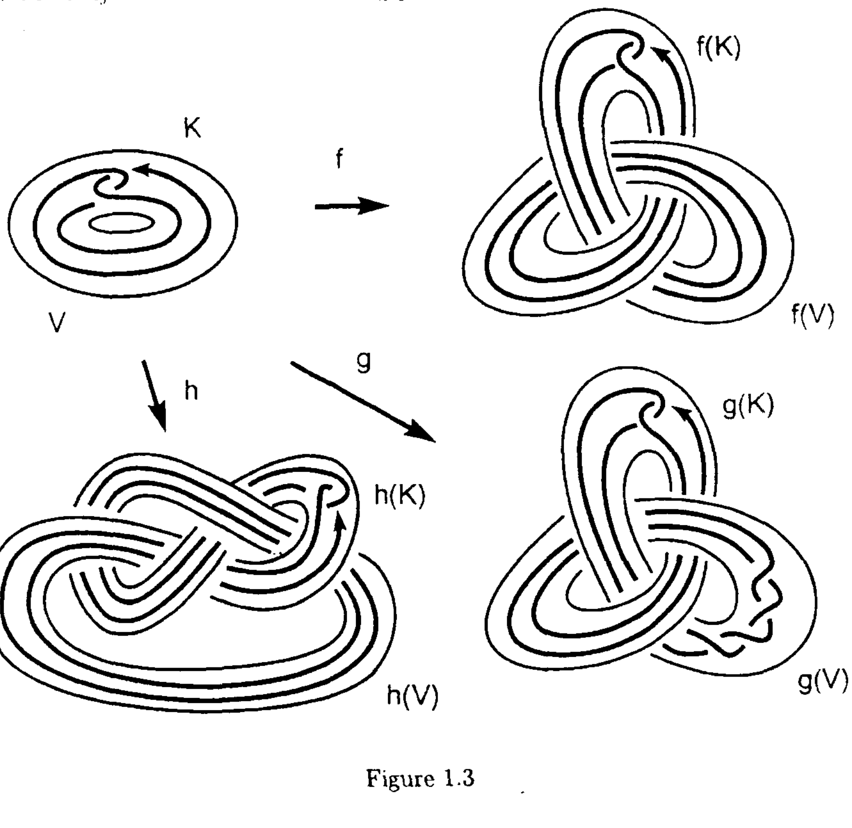


Figure 1.3

Problem 1.1. *Let (V, K) be a pattern and $f : V \to S^3$ an orientation preserving embedding such that $f(V)$ is knotted in S^3. Determine orientation preserving embeddings $g : V \to S^3$, up to isotopy, such that $g(K) \cong f(K)$.*

When $\mathrm{wrap}_V(K) = 1$, the operation "taking satellite" is the same as "taking product" (Figure 1.4).

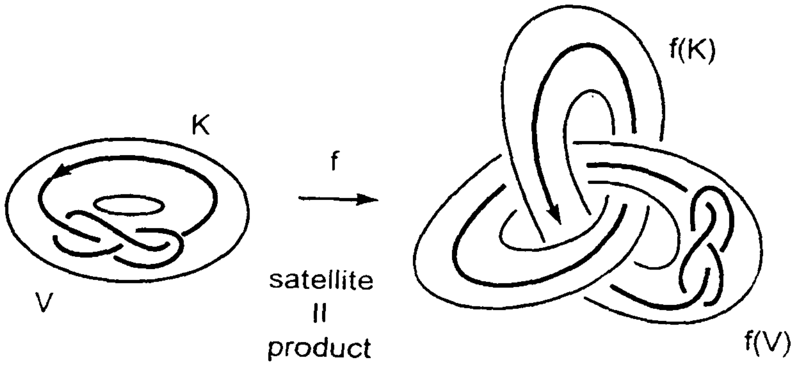


Figure 1.4

In this case, we have the following motivating result due to Schubert [17].

Theorem 1.2. **(Implication of Schubert's unique factorization theorem** [17]) *Let (V, K) be a pattern with $\mathrm{wrap}_V(K) = 1$ and $f, g : V \to S^3$ be two orientation preserving embeddings. Then $f(K) \cong g(K)$ if and only if $f(C_V) \cong g(C_V)$, where C_V denotes an oriented core of V.*

This is the result which we would like to generalize to any pattern. Before stating a result, we start with some examples.

A pattern (V, K) is said to be *symmetric* if (V, K) admits an orientation preserving homeomorphism $\psi : V \to V$ which satisfies $[\psi(C_V)] = -[C_V] \in H_1(V)$ and $\psi(K) = K$. Let s be the π-rotation along the axis L as shown in Figure 1.5. By the definition, for a symmetric pattern (V, K), K is null-homologous in V (or equivalently $\mathrm{wind}_V(K) = 0$).

The pattern (V, K) given by Figure 1.2 is symmetric. In fact, for some homeomorphism φ isotopic to the identity $\varphi \circ s$ gives the symmetry of (V, K).

For any symmetric pattern we can observe

Example 1.1. Let (V, K) be a symmetric pattern and $f : V \to S^3$ an orientation preserving embedding such that $f(C_V)$ is a non-invertible knot (i.e. $f(C_V) \not\cong$

$-f(C_V)$). Then for two embeddings f and $g = f \circ s$, we have $f(K) \cong g(K)$ and $f(C_V) \not\cong g(C_V)$. In particular f and g are not isotopic.

Proof. By the choice of g, $g(C_V) \cong -f(C_V)$. Hence $f(C_V) \not\cong g(C_V)$. Using the symmetry of (V, K) we can verify that $f(K) \cong g(K)$. □

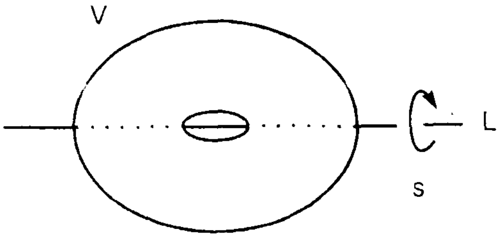


Figure 1.5

As a concrete example we may take the pattern given by Figure 1.2 and an embedding $f : V \to S^3$ so that $f(C_V)$ is the pretzel knot $K(3,\ 5,\ 7)$ (see Figure 1.6), which is known to be non-invertible [22].

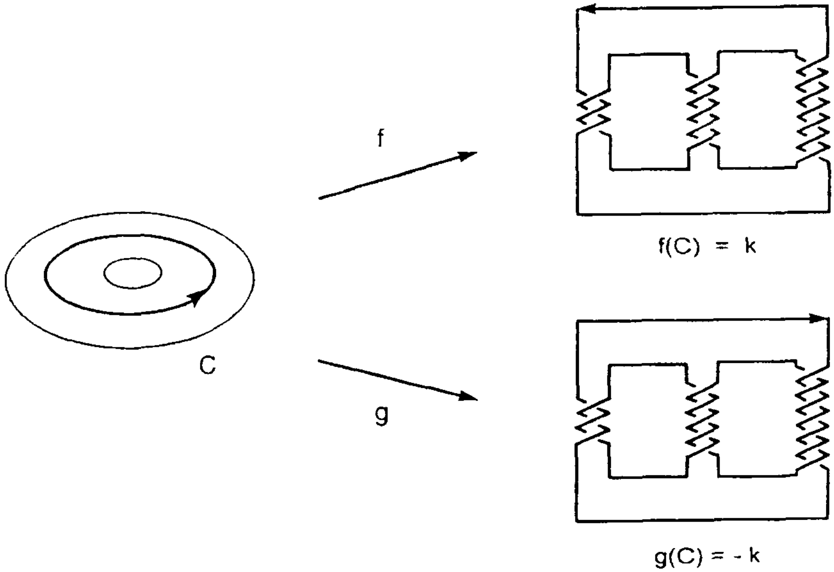
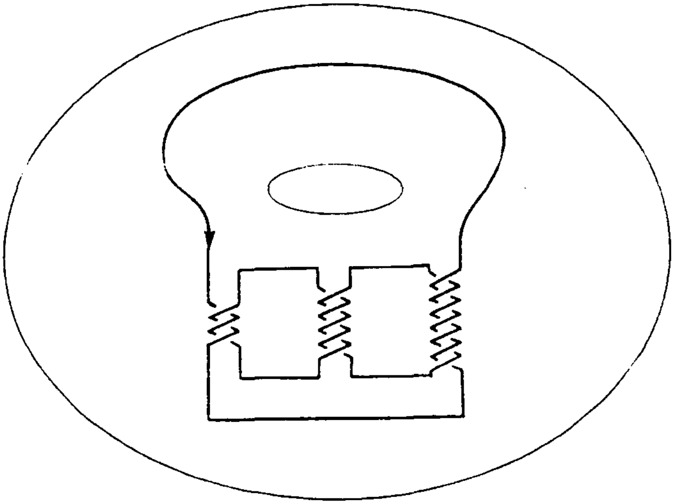


Figure 1.6

Example 1.2. Let (V, k) be a pattern such that $\mathrm{wrap}_V(k) = 1$ and $k \cong K(3,\ 5,\ 7)$ in S^3 (Figure 1.7). (For simplicity we assume that $[k] = [C_V] \in H_1(V)$.)



(V , k)

Figure 1.7

Let (V, K) be a pattern such that K is the untwisted double of k. Let f (resp. g) be an orientation preserving embedding from V into S^3 so that $f(C_V) \cong (-k)\sharp(-k)$ (resp. $g(C_V) \cong k\sharp(-k)$), see Figure 1.8.

Then for two embeddings f and g, we have $f(K) \cong g(K)$. But $f(C_V)$ and $g(C_V)$ cannot be equivalent even in the weakest sense.

Proof. We note that since $k \cong K(3,\ 5,\ 7)$ is non-invertible and non-amphicheiral, no two of k, $-k$, k^*, $-k^*$ are equivalent, where k^* denotes the mirror image of k [22]. In addition k is of genus one and so it is prime. Suppose that there is a homeomorphism of S^3 carrying $f(C_V)$ onto $\varepsilon g(C_V)$ ($\varepsilon = \pm 1$). Then we have $(-k)\sharp(-k) \cong (-\varepsilon k)\sharp(\varepsilon k)$ or $(-k)\sharp(-k) \cong (-\varepsilon k^*)\sharp(\varepsilon k^*)$. In any case $-k \cong k$ or $k \cong k^*$ must hold by Schubert's unique factorization theorem. This is a contradiction. Therefore $f(C_V)$ and $g(C_V)$ are not equivalent even in the weakest sense.

Let us prove $f(K) \cong g(K)$. First we note that $f(K)$ is the untwisted double of $(-k)\sharp(-k)\sharp k$ and $g(K)$ is the untwisted double of $k\sharp(-k)\sharp k \cong -((-k)\sharp(-k)\sharp k)$. Using the symmetry of the pattern given by Figure 1.2, we can observe that $f(K) \cong g(K)$. □

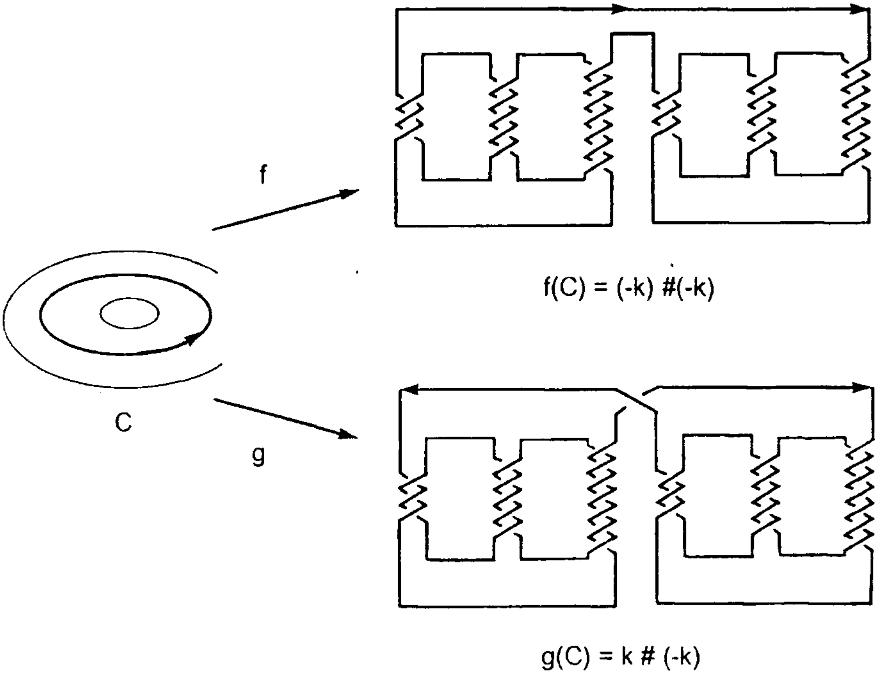


Figure 1.8

Example 1.2 shows that Theorem in [6] and hence also Corollary in [19] are not true.

The theorem below shows that the examples described above are worst that can happen. To state the result we prepare some terminologies.

Let C_V be an oriented core of a solid torus V in S^3. Then we choose a preferred meridian-longitude pair (m_V, ℓ_V) of V so that $[\ell_V] = [C_V] \in H_1(V)$ and $\ell k(m_V, C_V) = 1$, where $\ell k(\alpha, \beta)$ denotes the linking number of α and β. Let $f : V \to S^3$ be an orientation preserving embedding, then we adopt $f(C_V)$ as an oriented core of $f(V)$. This determines a preferred meridian-longitude pair $(m_{f(V)}, \ell_{f(V)})$ of $f(V)$ so that $[\ell_{f(V)}] = [f(C_V)] \in H_1(f(V))$ and $\ell k(m_{f(V)}, f(C_V)) = 1$. Then we have an expression $[f(\ell_V)] = [\ell_{f(V)}] + n[m_{f(V)}] \in H_1(\partial f(V))$ for some integer n. We define the *twist number* of the embedding $f : V \to S^3$ to be n and denote it by twist(f). Note that $[f(m_V)] = [m_{f(V)}]$. An orientation preserving embedding $f : V \to S^3$ is said to be *faithful* if twist$(f) = 0$.

Theorem 1.3 ([8]). *Let (V, K) be a pattern with* $\mathrm{wrap}_V(K) \geq 2$ *and* $f : V \to S^3$ *an orientation preserving embedding such that $f(V)$ is knotted in S^3. Let $g : V \to S^3$ be an orientation preserving embedding which satisfies $g(K) \cong f(K)$. Then $g(C_V) \cong f(C_V)$, or $f(C_V) \cong K_0 \sharp K_1$ and $g(C_V) \cong (-K_0) \sharp K_1$, where K_0 and K_1 are knots uniquely determined by the embedding f and the pattern (V, K). Furthermore in any case* $\mathrm{twist}(f) = \mathrm{twist}(g)$.

We remark that the above decomposition $f(C_V) \cong K_0 \sharp K_1$ does not depend on g. To make precise, we explain how we can determine the decomposition of $f(C_V)$ in the above theorem.

Let W be a solid torus in V. We say that W has the *property* $(*)$ if the following conditions are satisfied.

- W contains K in its interior,
- $\mathrm{wrap}_V(C_W) = 1$, where C_W is a core of W,
- C_W is not a core of V.

If there is no solid torus W in V satisfying the property $(*)$, then we put $K_0 = f(C_V)$ and define the decomposition of $f(C_V)$ to be $f(C_V) \cong K_0$.

Now let us assume that there is a solid torus W in V satisfying the property $(*)$. We say that the solid torus $W(\subset \mathrm{int}V)$ is $(*)$-*minimal* if there is no further solid torus $W'(\subset \mathrm{int}W)$ satisfying the property $(*)$ for W. Then by the uniqueness of the torus decomposition [4] [5], if there is a solid torus satisfying the property $(*)$ in V, then there exists a $(*)$-minimal solid torus W in V, unique up to isotopy.

Let W be a $(*)$-minimal solid torus in V. We choose an orientation of C_W so that C_W is homologous to C_V. Then we have a description $C_W \cong C_V \sharp k$ for some nontrivial knot k (see Figure 1.9).

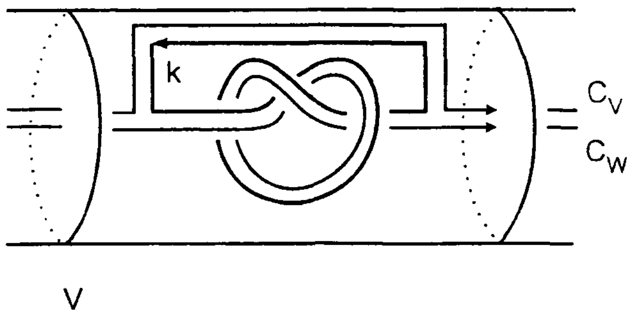


Figure 1.9

Let $k = k_1 \sharp \ldots \sharp k_n$ be a prime decomposition of k. First we delete all the invertible factors and all the pairs (k_i, k_j) with $k_i \cong k_j$. As a result we obtain $k_1 \sharp \ldots \sharp k_m$ (re-indexing if necessary). Then we put $K_1 = -(k_1 \sharp \ldots \sharp k_m)$. (Possibly K_1 is a trivial knot.) If $f(C_V)$ has an expression $f(C_V) \cong K_0 \sharp K_1$ for some knot K_0, then we define the decomposition of $f(C_V)$ to be $f(C_V) \cong K_0 \sharp K_1$. If $f(C_V)$ admits no such expression we put $K_0 = f(C_V)$ and define the decomposition of $f(C_V)$ to be $f(C_V) \cong K_0$ itself. It should be noted that the decomposition of $f(C_V)$ depends only on the pattern (V, K) and the embedding f.

Example 1.1 corresponds to the case where K_1 is trivial in Theorem 1.3. In Example 1.2, the decomposition of $f(C_V)$ is given by $f(C_V) \cong (-k) \sharp (-k)$ and this example shows that K_1 in Theorem 1.3 can be nontrivial.

Let $f, g : V \to S^3$ be orientation preserving embeddings. Then f and g are isotopic if and only if $f(C_V) \cong g(C_V)$ and $\text{twist}(f) = \text{twist}(g)$. Theorem 1.3, together with this fact, answers the question: How many embeddings (up to isotopy) can give equivalent satellite knots?

Corollary 1.4 ([8]). *Let (V, K) and f be as in Theorem 1.3. Then there is at most one orientation preserving embedding (up to isotopy) $g : V \to S^3$ which is not isotopic to f and $g(K) \cong f(K)$.*

If we assume further that $\text{wind}_V(K) \neq 0$, then we can improve Theorem 1.3 as follows.

Theorem 1.5 ([8]). *Let (V, K) be a pattern such that $\text{wrap}_V(K) \geq 2$ and K is homologically essential in V, and let $f : V \to S^3$ be an orientation preserving embedding such that $f(V)$ is knotted in S^3. If an orientation preserving embedding $g : V \to S^3$ satisfies $g(K) \cong f(K)$, then g is isotopic to f.*

2. Satellite knots obtained from a given embedding

Recall that to define a satellite knot we need two parameters: a pattern (V, K) and an embedding $f : V \to S^3$. In the previous section we consider a family of satellite knots obtained from the same pattern, i.e., we take an infinite family of embeddings of V into S^3 as parameters. On the contrary, in this section, we consider satellite knots obtained from the same embedding $V \to S^3$, i.e., we take patterns as parameters. In the following we consider unoriented knots in the oriented 3-sphere S^3. For two (unoriented) knots K_1 and K_2, we continue to write $K_1 \cong K_2$ to denote that K_1 and K_2 are ambient isotopic in S^3. For two patterns (V, K_1) and (V, K_2), if there exists an orientation preserving self-homeomorphism h of V sending *preferred-longitude* to $\pm$*preferred-longitude* which satisfies $h(K_1) = K_2$, then we write $(V, K_1) \sim (V, K_2)$. Furthermore if the homeomorphism h sends *preferred-longitude* to *preferred-longitude*, then we write $(V, K_1) \cong (V, K_2)$. $(V, K_1) \cong (V, K_2)$ if and only if K_1 and K_2 are ambient isotopic in V.

Theorem 2.1 ([12]). *Let* (V, K_i) $(i = 1, 2)$ *be a pattern. Suppose that* K_1 *is unknotted in* S^3 *and* $\mathrm{wind}_V(K_2) \neq 0$. *If* $f(K_1) \cong f(K_2)$ *in* S^3 *for some orientation preserving embedding* $f : V \to S^3$, *then* $(V, K_1) \sim (V, K_2)$ *holds.*

The next example shows the necessity of the condition "$\mathrm{wind}_V(K_2) \neq 0$".

Example 2.1. In Figure 2.1, K_1 is unknotted in S^3 and K_2 is knotted in S^3. However $\mathrm{wind}_V(K_2) = 0$. Figure 2.2 indicates an isotopy between $f(K_1)$ and $f(K_2)$.

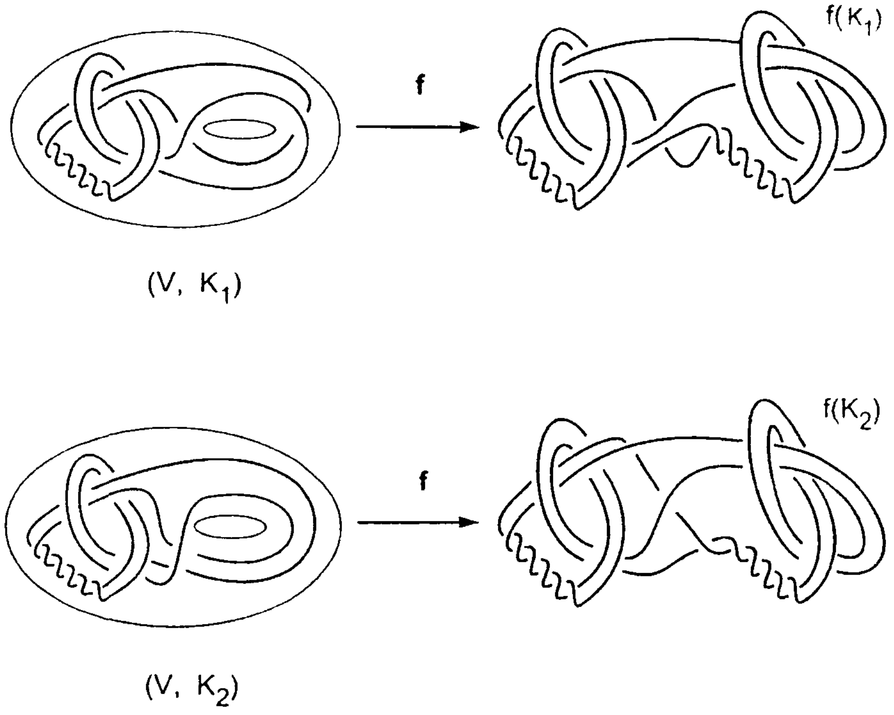


Figure 2.1

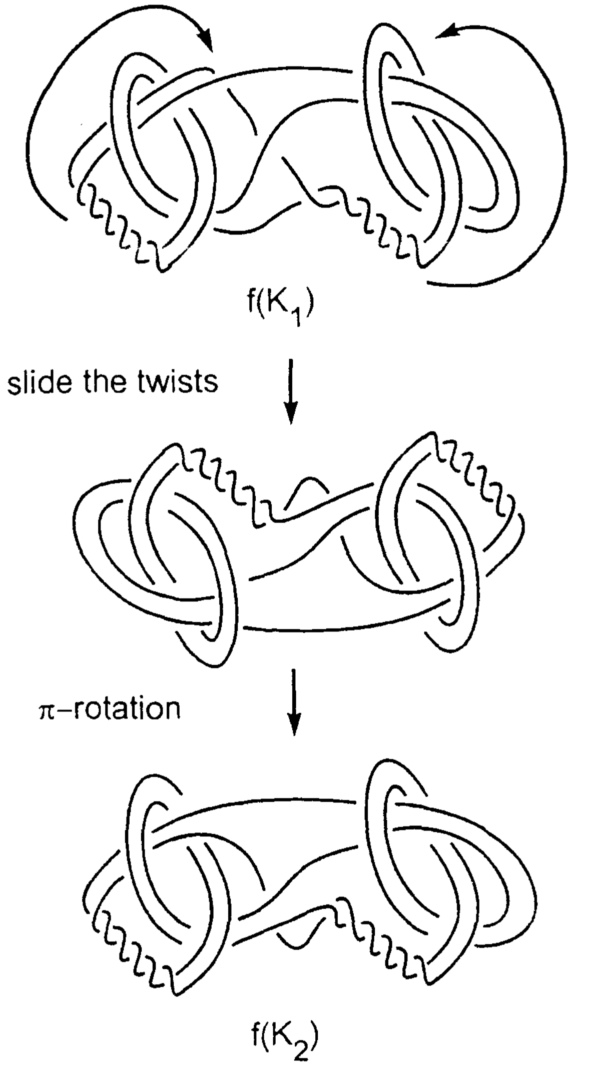


Figure 2.2

If K_1 is knotted, even when $\mathrm{wind}_V(K_2) \neq 0$, there is an example such that $(V, K_1) \not\cong (V, K_2)$ but $f(K_1) \cong f(K_2)$ in S^3, see Example 2.2.

Example 2.2.

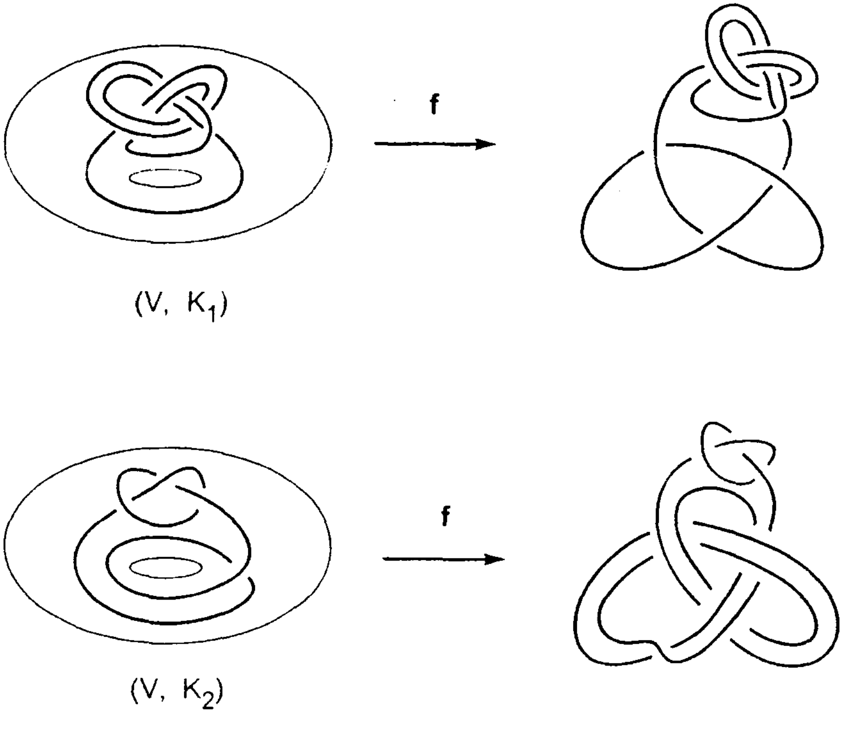


Figure 2.3

We then apply Theorem 2.1 to questions:

(1) Suppose that K_1 is unknotted and K_2 is knotted in S^3. Can $f(K_1)$ be ambient isotopic to $f(K_2)$ in S^3 for some embedding $f : V \to S^3$?

(2) Suppose that K_1 and K_2 are both unknotted in S^3. How are patterns (V, K_1) and (V, K_2) related if $f(K_1)$ and $f(K_2)$ are ambient isotopic in S^3 for some embedding $f : V \to S^3$?

We can answer the first question by

Corollary 2.2 ([12]). *Let (V, K_i) $(i = 1, 2)$ be a pattern. Suppose that K_1 is unknotted and K_2 is knotted in S^3 and* $\mathrm{wind}_V(K_2) \neq 0$. *Then for any embedding $f : V \to S^3$, $f(K_1) \not\cong f(K_2)$ in S^3.*

Proof. Assume that $f(K_1) \cong f(K_2)$ for some embedding $f : V \to S^3$. Then from Theorem 2.1, we have $(V, K_1) \sim (V, K_2)$. Extending the orientation preserving homeomorphism of V to that of S^3, we see that $K_1 \cong K_2$ in S^3, a contradiction. □

Example 2.1 shows the necessity of the condition "$\mathrm{wind}_V(K_2) \neq 0$" in Corollary 2.2.

As a special case of Theorem 2.1, we have the following which gives an answer to the second question.

Corollary 2.3 ([12]). *Let (V, K_i) be a pattern and K_i a trivial knot in S^3 $(i = 1, 2)$. Suppose that* $\mathrm{wind}_V(K_1) \neq 0$ *or* $\mathrm{wind}_V(K_2) \neq 0$. *If $f(K_1) \cong f(K_2)$ for some embedding $f : V \to S^3$, then $(V, K_1) \sim (V, K_2)$.*

Since $(V, K_1) \sim (V, K_2)$ implies $\mathrm{wind}_V(K_1) = \mathrm{wind}_V(K_2)$ and $\mathrm{wrap}_V(K_1) = \mathrm{wrap}_V(K_2)$, we have the following.

Corollary 2.4 ([12]). *Suppose that K_i is a trivial knot contained in a standardly embedded solid torus V in S^3 $(i = 1, 2)$.*

(1) *If* $\mathrm{wind}_V(K_1) \neq \mathrm{wind}_V(K_2)$, *then $f(K_1) \not\cong f(K_2)$ in S^3 for any embedding $f : V \to S^3$.*

(2) *When* $\mathrm{wind}_V(K_1) = \mathrm{wind}_V(K_2) \neq 0$, *if* $\mathrm{wrap}_V(K_1) \neq \mathrm{wrap}_V(K_2)$, *then $f(K_1) \not\cong f(K_2)$ in S^3 for any embedding $f : V \to S^3$.*

In the case where $\mathrm{wind}_V(K_1) = \mathrm{wind}_V(K_2) = 0$, the situation is quite different.

Theorem 2.5 ([12]). *For any faithful embedding $f : V \to S^3$ (i.e.,* $\mathrm{twist}(f) = 0$), *there exist patterns (V, K_1) and (V, K_2) which satisfy*

(1) *both K_1 and K_2 are unknotted in S^3,*

(2) $\mathrm{wind}_V(K_1) = \mathrm{wind}_V(K_2) = 0$, $(V, K_1) \not\sim (V, K_2)$, *and*

(3) *$f(K_1) \cong f(K_2)$ in S^3.*

Proof. For the given faithful embedding f, actually we can construct required patterns as follows; the construction is due to Makoto Sakuma.

First let us consider a 3-components Brunnian link $L = k \cup L_1 \cup L_2$ depicted in Figure 2.4.

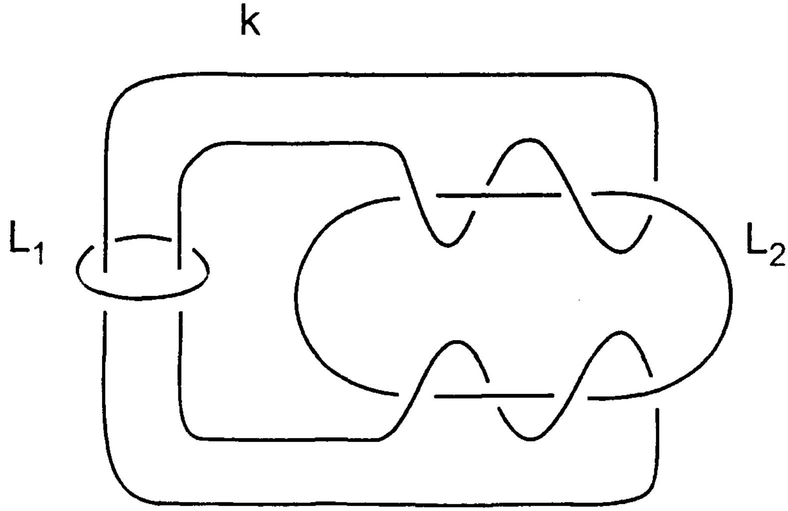


Figure 2.4

Let (m_i, ℓ_i) be a preferred meridian-longitude pair of L_i $(i = 1,2)$. Let t be a knot ambient isotopic to $f(C)$, where C denotes a core of V, and (m, ℓ) a preferred meridian-longitude pair of t. Removing a tubular neighborhood $N(L_i)$ of L_i and gluing the knot exterior $E(t) = S^3 - \text{int}N(t)$ so that $m_i = \ell$ and $\ell_i = m$, we obtain $S^3 \cong (S^3 - \text{int}N(L_i)) \cup_{\substack{m_i = \ell \\ \ell_i = m}} E(t)$ and a new knots K_{3-i} and $\tilde{L}_{3-i}$ as the images of k and L_{3-i} respectively, for $i = 1, 2$. It is easy to see that both K_{3-i} and $\tilde{L}_{3-i}$ are unknotted in S^3. Hence by putting $V = S^3 - \text{int}N(\tilde{L}_{3-i})(\supset K_{3-i})$, we have a pattern (V, K_{3-i}) with $\text{wind}_V(K_{3-i}) = 0$. In this way we obtain two patterns (V, K_1) and (V, K_2). By the construction, for the faithful embedding $f : V \to S^3$, $f(K_1) \cong f(K_2)$ in S^3. In fact, roughly speaking, $f(K_1)$ and $f(K_2)$ can be described as the knot obtained from k in Figure 2.4 by simultaneously replacing neighborhoods of disks bounded by L_1 and L_2 by tubes knotted according to the given knot t.

We can prove $(V, K_1) \not\cong (V, K_2)$ by showing that $\text{wrap}_V(K_1) = 2$ and $\text{wrap}_V(K_2) = 4$. For more details, see [12]. □

This result can be generalized to

Corollary 2.6 ([12]). *For any knot K in S^3 and any faithful embedding $f : V \to S^3$, there exist patterns (V, K_1) and (V, K_2) which satisfy*

(1) *$K_i \cong K$ in S^3 for $i = 1, 2$,*

(2) $\text{wind}_V(K_1) = \text{wind}_V(K_2) = 0$, $(V, K_1) \not\cong (V, K_2)$, *and*

(3) $f(K_1) \cong f(K_2)$ *in* S^3.

Proof. Let (V, k_1) and (V, k_2) be the patterns given by Theorem 2.5 depending on the embedding $f : V \to S^3$. Since each k_i is unknotted in S^3, we can locally replace an unknotted arc of k_i by a knotted arc (with a suitable direction) so that the result K_i represents K in S^3. Then it follows from the choice of (V, k_i) that (V, K_1) and (V, K_2) are the desired patterns. □

3. Knots obtained from trivial knots by twisting

In previous sections, we assume that $f(V)$ is knotted in S^3. In what follows we consider the case where $f(V)$ is also unknotted in S^3. So we may assume that $f(V) = V$ and $f : V \to V$ is a twisting homeomorphism of V with $f(\mu) = \mu$ and $f(\lambda) = \lambda + n\mu$, where (μ, λ) is a preferred meridian-longitude pair of V. We denote the image of K by K_n. (See Figure 3.1.)

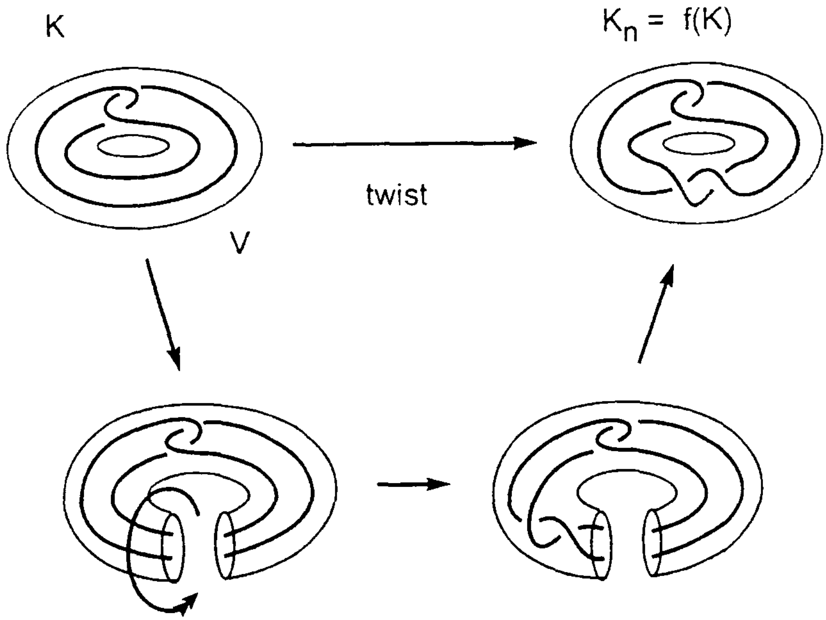


Figure 3.1

When the wrapping number of K in V is zero, this operation does not affect the knot types, so we always assume that $\mathrm{wrap}_V(K) \geq 2$ in the following.

If the original knot K is a trivial knot in S^3, then we call the resulting knot K_n a *twisted knot*.

In this section we consider the possibility obtaining knots of special kinds from trivial knots by twisting.

3.1. When can a twisted knot be a trivial knot?

In such a special situation as indicated in the title, applying Gabai's result [1], we can deduce the following.

Theorem 3.1 ([9], [7]). *A twisted knot K_n ($n \neq 0$) is knotted in S^3, except for the case as in Figure 3.2.*

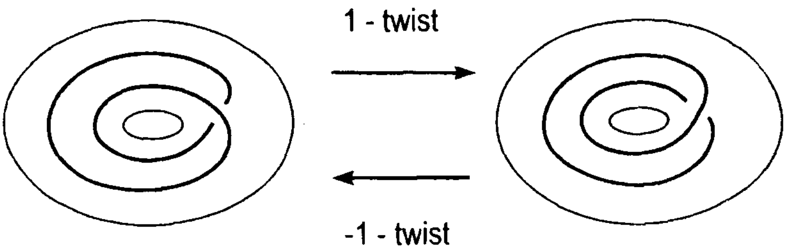


Figure 3.2

This result can be regarded as an answer to a very special case of the following conjecture.

Conjecture 3.2. *Let K be a knot in a standardly embedded solid torus V in S^3. A knot K_n obtained from K by n-twist ($n \neq 0$) cannot be ambient isotopic to K, except for the case as in Figire 3.3.*

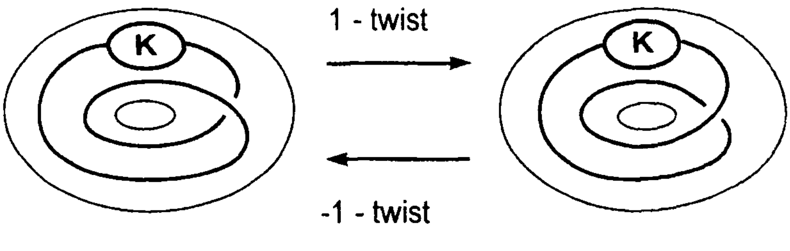


Figure 3.3

3.2. When can a twisted knot be a composite knot?

We start with a motivating result proved by Scharlemann [15].

Theorem 3.3 **([15], see also [23]).** *A crossing change on a trivial knot cannot produce a composite knot (i.e., a knot of unknotting number one is prime).*

Note that a crossing change on a knot K can be accomplished by a ± 1-twist with $\mathrm{wrap}_V(K) = 2$ for some V, and Theorem 3.3 was generalized to the following.

Theorem 3.4 **([16]).** *Let V be a standardly embedded solid torus in S^3 and K a knot in V with $\mathrm{wrap}_V(K) = 2$. Then a twisted knot K_n cannot be a composite knot for any integer n.*

In connection with this, Mathieu [9] proposed the question:

Question 3.5 **([9]).** *Can we have a composite twisted knot?*

Theorem 3.4 shows the impossibility in the case $\mathrm{wrap}_V(K) = 2$. On the contrary, we can answer this question in positive by constructing the following concrete example.

Example 3.1 **([14]).** Let (V, K) be a pattern depicted in Figure 3.4. Then K_1 is a product of the $(2,3)$-torus knot and the $(2,5)$-torus knot.

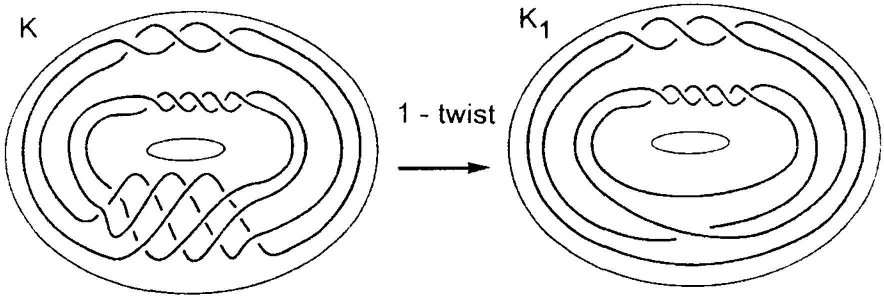


Figure 3.4

In this example $\mathrm{wrap}_V(K) = 4$. Later Ohyama also foud an interesting and simpler example with $\mathrm{wrap}_V(K) = 3$ such that K_1 is a prduct of the $(2,3)$-torus knot and the figure eight knot. We can find other examples of composite twisted knots in [20], [11] and [2]. There is an excellent account of examples of composite twisted knots in [2]. It should be noted that all the examples are ± 1-twist, and it was conjectured in [13] that a twisted knot can be a composite knot only for one integer $n \in \{1, -1\}$. In [21], Teragaito proved that if a twisted knot K_n is a composite knot then $|n| \leq 2$ applying a combinatorial technique developed by Gordon and Luecke. Later Goodman-Strauss [2], Hayashi and the author [3] independently proved the following.

Theorem 3.6 ([2], [3]). *If a twisted knot K_n is a composite knot, then $n = \pm 1$.*

Goodman-Strauss [2] shows further that K_1 and K_{-1} cannot both be composite knots.

But the following question is still open.

Question 3.7. *Is the number of prime factors of $K_{\pm 1} \leq 2$?*

Compare this with the following well-known question: Is the number of prime factors of a manifold obtained by Dehn surgery on a knot in S^3 is less than or equal to 2?

3.3. When can a twisted knot be a torus knot?

Let us start with a well-known example. Let K be a $(\pm 1, q)$-cable of a core of a standardly embedded solid torus V in S^3. Then K is a trivial knot in S^3 and K_n is a torus knot (which is a trivial knot again if $q = 2$ and $n = \mp 1$) for any integer $n (\neq 0)$. We refer such an example as *trivial example* (see Figure 3.5).

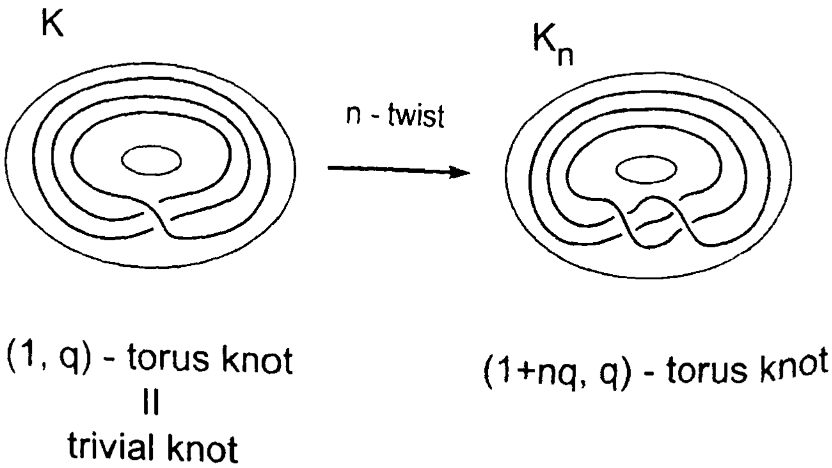


Figure 3.5

As a nontrivial example, we have the following.

Example 3.2.

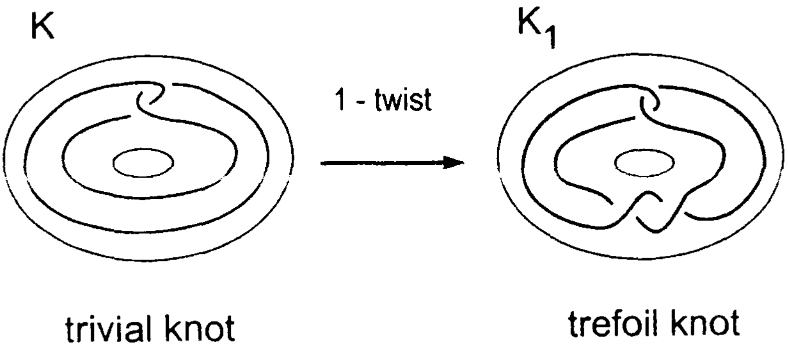


Figure 3.6

Theorem 3.8. *If a twisted knot K_n $(n \neq 0)$ is a torus knot, then except for trivial examples $n = \pm 1$.*

This result is an implicit corollary of the joint work with Miyazaki [10, Theorem 1.2] about Seifert fibring surgery on knots in solid tori.

Theorem 3.9 ([10]). *Let J be a knot in a solid torus W such that J is not contained in a 3-ball in W. Suppose that a manifold $W(J;\gamma)$ obtained from W by γ-surgery on J is Seifert fibred. Then one of the following holds.*

(1) *J is a core of W or a cable of a 0-bridge braid in W.*

(2) *γ is integral (i.e., a representative of γ intersects a meridian of J exactly once).*

Proof of Theorem 3.8. Let J be a core of the complementary solid torus $S^3-\text{int}V$. We note that the twisted knot K_n can be obtained from K by $-\frac{1}{n}$-surgery on J. Since K is a trivial knot in S^3, $W = S^3-\text{int}N(K)$ is a solid torus, which contains J in its interior. If K_n is a torus knot, then $E(K_n) \cong W(J; -\frac{1}{n})$ is a Seifert fibred manifold over the disk with two exceptional fibres. Applying Theorem 3.9, we have the following possibilities:

(1) J is a core of W or a cable of a 0-bridge braid in W.

(2) $n = \pm 1$.

Now we suppose that (1) happens. If J is a core of W, then we have $\text{wrap}_V(K) = 1$, a contradiction. If J is a cable of a 0-bridge braid in W, then it turns out that K is also a cable of a 0-bridge braid in V. Assume first that the 0-bridge braid is not a core of V. Then since K is unknotted in S^3, the 0-bridge braid is a $(\pm 1, q)$-cable of a core of V $(q \geq 2)$. Hence K_n is a cable of a $(\pm 1 + nq, q)$-cable of a core of W. The twisted knot K_n can be a torus knot only when $n = \mp 1$ and $q = 2$, otherwise K_n has a nontrivial companion. Next assume that the 0-bridge braid is a core of W, then K is a $(\pm 1, q)$-cable of a core of V. In this case we have exactly a trivial example. This completes the proof of Theorem 3.8. □

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