X is *locally Euclidean* if for all $x \in X$, there exists U open such that $x \in U$, and there exists a homeomorphism $f : U \to f(U) \subset \mathbf{R}^m$ where $f(U)$ is open in $\mathbf{R}^m$.

Ex: $(0, 1)$ is locally Euclidean, but $[0, 1]$ is NOT locally Euclidean.

X is an $m-$*manifold* if
(1) X is locally Euclidean
(2) X is T_2
(3) X 2nd countable.

A 1-manifold is a *curve* (ex: the circle S^1)

A 2-manifold is a *surface* (ex: the torus T^2, the projective plane $\mathbf{R}P^2$, the Klein bottle, $\mathbf{R}P^2 \# \mathbf{R}P^2$).

The *support* of $\phi : X \to \mathbf{R} = \overline{\phi^{-1}(\mathbf{R} - \{0\})}$.

I.e., $x \notin$ support ϕ iff there exists U open such that $x \in U$ and $\phi(U) = \{0\}$.

Ex:

Let $\{U_1, ..., U_n\}$ be a finite indexed open cover of X.

An indexed family of continuous functions

$$\phi_i : X \to [0, 1]$$

is a *partition of unity dominated by* $\{U_1, ..., U_n\}$ if
1) support $\phi_i \subset U_i$ for all i.
2) $\Sigma_{i=1}^n \phi_i(x) = 1$ for all x.

Ex: $\phi_i : \mathbf{R} \to [0, 1], \phi_i(x) = \frac{1}{2}$ is a partition of unity dominated by $U_i = \mathbf{R}$, $i = 1, 2$

Note: partition of unity for an arbitrary open cover will be defined in section 41 (one more condition, which finite covers automatically satisfy, will be needed).

Section 39:

A collection $\mathcal{A}$ of subsets of X is *locally finite* if for all $x \in X$, there exists U open such that $x \in U$ and U intersects only finitely many elements of $\mathcal{A}$

Ex: $\mathcal{A} = \{(n, n+2) \mid n \in \mathbf{Z}\}$ is locally finite.

Ex: $\mathcal{C} = \{(n, n+2) \mid n \in \mathbf{Z}_+\}$ is locally finite.

Ex: $\mathcal{D} = \{(0, n) \mid n \in \mathbf{Z}_+\}$ is NOT locally finite.

Ex: A finite collection of sets is locally finite.

The indexed family $\{A_\alpha \mid \alpha \in J\}$ is a *locally finite indexed family* in X if for all $x \in X$, there exists U open such that $x \in U$ and U intersects A_α for only finitely many α.

Ex: If $A_i = \mathbf{R}$ for all $i \in \mathbf{Z}$, then $\{A_i \mid i \in \mathbf{Z}\}$ is NOT a locally finite indexed family in X, but $\{A_i \mid i \in \mathbf{Z}\}$, as a collection of set(s), is locally finite (since it contains only one set).

A collection $\mathcal{A}$ of subsets of X is *countably locally finite* if $\mathcal{A}$ can be written as a countable union of collections $\mathcal{A}_n$, each of which is locally finite.

Ex: $\mathcal{D} = \{(-n, n) \mid n \in \mathbf{Z}_+\}$ is countable locally finite. Let $\mathcal{D}_k = \{(-n, n) \mid n \in [k, k+2]\}$, $k \in 2\mathbf{Z}_+$.

Note $\mathcal{D} = \cup_{k \in 2\mathbf{Z}_+} \mathcal{D}_k$ and $\mathcal{D}_k$ is locally finite since it's finite.

Let $\mathcal{A}$ be a collection of subsets of X. A collection $\mathcal{B}$ of subsets of X is a *refinement* of $\mathcal{A}$ if for all $B \in \mathcal{B}$, there exists $A \in \mathcal{B}$ such that $B \subset A$.

If the elements of $\mathcal{B}$ are open then $\mathcal{B}$ is an *open refinement* of $\mathcal{A}$.
If the elements of $\mathcal{B}$ are closed then $\mathcal{B}$ is a *closed refinement* of $\mathcal{A}$.

A simply ordered set X is *well ordered* if every nonempty subset of X has a smallest element (ie $A \subset X$, $A \neq \emptyset$ implies $min(A)$ exists and $min(A) \in A$).

Ex: $\mathbf{Z}$ is NOT well-ordered.
Ex: $\mathbf{Z}_+$ is well-ordered
Ex: $\mathbf{R}_+$ is NOT well-ordered.

The Well-ordering theorem: If X is a set, there exists an order relation on X that is well-ordered.