

26. Compact Spaces

A family of sets, $\mathcal{F} = \{F_\alpha \mid \alpha \in A\}$, **covers** the set S if

$$S \subset \bigcup_{\alpha \in A} F_\alpha$$

\mathcal{F} is said to be a cover of S .

$\mathcal{F}_1 = \{ (x - 1, x + 1) \mid x \in S \}$ is a cover of S since

$$S \subset \bigcup_{x \in S} (x - 1, x + 1)$$

If $S \neq \emptyset$, take $x_0 \in S$

$\mathcal{F}_2 = \{ (x_0 - r, x_0 + r) \mid r > 0 \}$ is a cover of S since

$$S \subset \bigcup_{r > 0} (x_0 - r, x_0 + r)$$

\mathcal{F}' is a **subcover** of \mathcal{F} if $\mathcal{F}' \subset \mathcal{F}$ and \mathcal{F}' covers S .

\mathcal{F}' is a **finite subcover** of \mathcal{F} (or a finite subfamily of \mathcal{F}) if \mathcal{F}' is a subcover of \mathcal{F} and \mathcal{F}' is finite.

$\mathcal{F} = \{ (\frac{1}{n}, 1) \mid n = 1, 2, 3, \dots \}$ is a cover of $(0, 1)$ since $(0, 1) \subset \bigcup_{n=1}^{\infty} (\frac{1}{n}, 1)$.

$\mathcal{F}' = \{ (\frac{1}{n}, 1) \mid n = 5, 6, 7, \dots \}$ is a subcover of \mathcal{F} since $\mathcal{F}' \subset \mathcal{F}$ and $(0, 1) \subset \bigcup_{n=5}^{\infty} (\frac{1}{n}, 1)$.

Does there exist a finite subcover?

Defn: A family of sets, $\mathcal{F} = \{F_\alpha \mid \alpha \in A\}$, is an **open cover** of S if \mathcal{F} covers S and if F_α is open for all $\alpha \in A$.

Defn: A space S is **compact** if every open cover of S has a finite subcover.

Lemma: If X is finite, then X is compact.

Lemma 26.1: Let Y be a subspace of X . Every cover of Y consisting of open sets in X has a finite subcover if and only if every cover of Y consisting of open sets in Y has a finite subcover

Thm 26.2: Every closed subspace of a compact space is compact.

Lemma 26.4: If Y is a compact subspace of the Hausdorff space X , and $x_0 \notin Y$, then there exist disjoint open sets U and V of X such that $x_0 \in U$ and $Y \in V$.

Thm 26.3: Every compact subspace of a Hausdorff space is closed.

Thm 26.5: The image of a compact space under a continuous map is compact.

Thm 26.6: If $f : X \rightarrow Y$ is continuous and a bijection and if X is compact and Y is Hausdorff, then f is a homeomorphism.

Note: $f : [0, 1) \rightarrow \{(x, y) \mid x^2 + y^2 = 1\}$, $f(x) = e^{2\pi i x}$ is continuous and a bijection, but f^{-1} is NOT continuous.

Note: $f : \{1, 2\} \rightarrow \{1, 2\}$, $f(n) = n$ is a bijection. $X = \{1, 2\}$ is compact since X is finite.

If X has the _____ topology and Y has the _____ topology, then f is continuous, but not a homeomorphism.

$Y = \{1, 2\}$ is not _____.

Lemma 26.8 (The tube lemma). Suppose N is an open set in $X \times Y$ where Y is compact. If there exists an $x_0 \in X$ such that $x_0 \times Y \subset N$, then there exists an open set W such that $x_0 \in W$ and $W \times Y \subset N$.

Thm 26.7: The product of finitely many compact spaces is compact.

Thm 37.3 (Tychonoff theorem). An arbitrary product of compact spaces is compact in the product topology.