22. The Quotient Topology

Defn: Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is a **quotient map** if U is open in Y if and only if $p^{-1}(U)$ is open in X.

Defn: $C \subset X$ is **saturated** with respect to p if $p^{-1}(\{y\}) \cap C \neq \emptyset$ implies $p^{-1}(\{y\}) \subset C$.

That is, C is saturated if there exists a set $D \subset Y$ such that $C = p^{-1}(D)$.

That is, C is saturated if $C = p^{-1}(p(C))$

Lemma: $p : X \to Y$ is a quotient map if and only if p is continuous and p maps saturated open sets of X to open sets of Y.

Defn: $f : X \to Y$ is an **open map** if for every open set U of X, $f(U)$ is open in Y.

Defn: $f : X \to Y$ is a **closed map** if for every closed set A of X, $f(A)$ is closed in Y.

Lemma: An open map is a quotient map. A closed map is a quotient map. There exist quotient maps which are neither open nor closed.

Defn: Let X be a topological spaces and let A be a set; let $p : X \to Y$ be a surjective map. The **quotient topology** on A is the unique topology on A which makes p a quotient map.

Defn: A partition, X^*, of a set X is a collection of disjoint subsets of X whose union is X. I.e., $X^* = \{C_\alpha \mid \alpha \in A\}$, $X^* \subset P(X)$, the power set on X, $C_\alpha \cap C'_\alpha = \emptyset$, for all $\alpha, \alpha' \in A$, and $X = \cup_{\alpha \in A} C_\alpha$.

Define $p : X \to X^*$, $p(x) = C_\alpha$ if $x \in C_\alpha$. The quotient topology induced by p on X^* is the **quotient space** of X. X^* is called the **identification space** or decomposition space of X.

Thm 22.2: Let $p : X \to Y$ be a quotient map. Let $g : X \to Z$ be a map with is constant on each set $p^{-1}(y)$ for all $y \in Y$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. f is continuous if and only if g is continuous. f is a quotient map if and only if g is a quotient map.

Cor 22.3: Let $g : X \to Z$ be a surjective continuous map. Let $X^* = \{g^{-1}(\{z\} \mid z \in Z\}$ with the quotient topology.
(a.) The map g induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism if and only if g is a quotient map.
(b.) If Z is Hausdorff, so in X^*.