20. The Metric Topology

Defn: Suppose \( d : X \times X \to R \). Then \( d \) is a metric on \( S \) if \( d \) satisfies the following conditions.

1.) \( d(x, y) \geq 0 \) for all \( (x, y) \in X \times X \);
\( d(x, y) = 0 \) if and only if \( x = y \).

2.) \( d(x, y) = d(y, x) \) for all \( (x, y) \in X \times X \).

3.) \( d(x, z) \leq d(x, y) + d(y, z) \forall x, y, z \in X \).

Example 1 (the euclidean metric on \( R^n \)):
\[
d_1(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}
\]
where \( x = (x_1, x_2, ..., x_n) \) and \( y = (y_1, y_2, ..., y_n) \).

Example 2 (the square metric on \( R^n \)):
\[
\rho(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|
\]
where \( x = (x_1, x_2, ..., x_n) \) and \( y = (y_1, y_2, ..., y_n) \).

Example 3 (the discrete metric on \( X \)):
\[
d_3(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}
\]

Example 4: Let \( C = \) set of all continuous real-valued functions on \([0, 1]\).
\[
d_4(f, g) = \max \{|f(x) - g(x)| \mid x \in [0, 1]\}.
\]

Defn: \( B_d(p, r) = \{ x \in X \mid d(p, x) < r \} \)

Defn: If \( d \) is a metric, then
\[
\{ B_d(p, r) \mid , p \in X, r > 0 \}
\]
is a basis for the metric topology on \( X \) induced by \( d \).

Lemma: \( U \) is open in the metric topology on \( X \) induced by \( d \) if for every \( y \in U \), there exists an \( r > 0 \) such that \( B_d(y, r) \subset U \).

Defn: If \( X \) is a topological space, \( X \) is said to be metrizable if there exists a metric \( d \) on \( X \) which induces the topology on \( X \). A metric space is a metrizable space \( X \) together with a specific metric \( d \) that gives the topology on \( X \).
Defn: Let $X$ be a metric space with metric $d$. A subset $A$ of $X$ is **bounded** if there exists a number $M$ such that $d(a_1, a_2) \leq M$ for every $a_1, a_2 \in A$. If $A$ is bounded and nonempty, the diameter of $A = \text{diam } A = \sup \{d(a_1, a_2) \mid a_1, a_2 \in A\}.$

Note that boundedness is not a topological property.

Thm 20.1: Let $X$ be a metric space with metric $d$. Define $\overline{d} : X \times X \to \mathbb{R}$ by

$$\overline{d}(x, y) = \min\{d(x, y), 1\}.$$

Then $\overline{d}$ is a metric that induces the same topology as $d$.

Defn: The metric $\overline{d}$ is called the **standard bounded metric** corresponding to $d$.

Lemma 20.2: Let $d$ and $d'$ be two metrics on $X$; let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each $x \in X$ and each $\epsilon > 0$, there exists a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$.

Corollary $\mathcal{T}'$ is finer than $\mathcal{T}$ if there exists a $k > 0$ such that for all $x, y \in X$:

Thm 20.3: The topologies on $\mathbb{R}^n$ induced by the euclidean metric $d$ and the square metric $\rho$ are the same as the product topology on $\mathbb{R}^n$. 

21 22