20. The Metric Topology

Defn: Suppose $d : X \times X \to R$. Then d is a **metric** on S if d satisfies the following conditions.

1.) $d(x, y) \geq 0$ for all $(x, y) \in X \times X$;
$d(x, y) = 0$ if and only if $x = y$.

2.) $d(x, y) = d(y, x)$ for all $(x, y) \in X \times X$.

3.) $d(x, z) \leq d(x, y) + d(y, z) \forall x, y, z \in X$.

Example 1 (**the euclidean metric on** R^n):

$$d_1(\mathbf{x}, \mathbf{y}) = \sqrt{\Sigma_{i-1}^{n}(x_i - y_i)^2}$$

where $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$.

Example 2 (**the square metric on** R^n):

$$\rho(\mathbf{x}, \mathbf{y}) = max_{\{1 \leq i \leq n\}} |x_i - y_i|$$

where $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$.

Example 3 (**the discrete metric on** X):

$$d_3(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases} .$$

Example 4: Let C = set of all continuous real-valued functions on [0, 1].

$$d_4(f, g) = max\{|f(x) - g(x)| \mid x \in [0, 1]\}.$$

Defn: $B_d(p, r) = \{x \in X \mid d(p, x) < r\}$

Defn: If d is a metric, then

$$\{B_d(p, r) \mid , p \in X, r > 0\}$$

is a basis for the **metric topology** on X induced by d.

Lemma: U is open in the metric topology on X induced by d if for every $y \in U$, there exists an $r > 0$ such that $B_d(y, r) \subset U$.

Defn: If X is a topological space, X is said to be **metrizable** if there exists a metric d on X which induces the topology on X. A **metric space** is a metrizable space X together with a specific metric d that gives the topology on X.

Defn: Let X be a metric space with metric d. A subset A of X is **bounded** if there exists a number M such that $d(a_1, a_2) \leq M$ for every $a_1, a_2 \in A$. If A is bounded and nonempty, the diameter of $A =$

$$\text{diam } A = sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}.$$

Note that boundedness is not a topological property.

Thm 20.1: Let X be a metric space with metric d. Define $\overline{d} : X \times X \to R$ by

$$\overline{d}(x, y) = min\{d(x, y), 1\}.$$

Then $\overline{d}$ is a metric that induces the same topology as d.

Defn: The metric $\overline{d}$ is called the **standard bounded metric** corresponding to d.

Lemma 20.2: Let d and d' be two metrics on X; let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce, respectively. Then $\mathcal{T}'$ is finer that $\mathcal{T}$ if and only if for each $x \in X$ and each $\epsilon > 0$, there exists a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$.

Corollary $\mathcal{T}'$ is finer that $\mathcal{T}$ if there exists a $k > 0$ such that for all $x, y \in X$:

Thm 20.3: The topologies on R^n induced by the euclidean metric d and the square metric ρ are the same as the product topology on R^n.