19. The Product Topology.

Defn: Let J be an index set. Given a set X, a **J-tuple** of elements of X is a function $\mathbf{x} : J \to X$. The **αth coordinate of $\mathbf{x}$** $= x_\alpha = \mathbf{x}(\alpha)$.

Defn: Let $X = \cup_{\alpha \in J} A_\alpha$.

$\Pi_{\alpha \in J} A_\alpha$= the **Cartesian product** of $\{A_\alpha\}_{\alpha \in J}$
$= \{(x_\alpha)_{\alpha \in J} \mid \; x_\alpha \in A_\alpha \text{ for each } \alpha \in J\}$.

That is, it is the set of all functions
$\mathbf{x} : J \to \cup_{\alpha \in J} A_\alpha$ such that $\mathbf{x}(\alpha) \in A_\alpha \; \forall \alpha \in J$.

Defn: The **box topology** on $\Pi_{\alpha \in J} X_\alpha$ is the topology generated by the basis

$$\{\Pi_{\alpha \in J} U_\alpha \mid U_\alpha \text{ open in } X_\alpha\}.$$

Defn: Let $\mathcal{S}_\alpha = \{\pi_\alpha^{-1}(U) \mid U \text{ open in } X_\alpha\}$
The **product topology** on $\Pi_{\alpha \in J} X_\alpha$ is the topology generated by the subbasis $\mathcal{S} = \cup_{\alpha \in J} \mathcal{S}_\alpha$.

Thm 19.1, 2: Comparison of box and product topologies. Let $\mathcal{B}_\alpha$ be a basis for X_α

Basis for the box topology: $\{\Pi U_\alpha \mid U_\alpha \text{ open in } X_\alpha\}$
or $\{\Pi B_\alpha \mid B_\alpha \in \mathcal{B}_\alpha\}$

Basis for the product topology:

$\{\Pi U_\alpha \mid U_\alpha \text{ open in } X_\alpha,$
$U_\alpha = X_\alpha \text{ for all but finitely many } \alpha\}$

or $\{\Pi B_\alpha \mid B_{\alpha_i} \in \mathcal{B}_{\alpha_i}, \; i = 1, ..., n,$
$B_\alpha = X_\alpha \text{ for } \alpha \neq \alpha_i, \; i = 1, ..., n\}$

Hence box topology is finer then the product topology

Thm 19.3: Let A_α be a subspace of X_α. Then ΠA_α is a subspace of ΠX_α if both products are given the box topology or if both products are given the product topology.

Thm 19.4: If X_α is Hausdorff for all α then ΠX_α is Hausdorff in both the box and product topologies.

Thm 19.5: $\Pi \overline{A_\alpha} = \overline{\Pi A_\alpha}$ in both the box and product topologies.

Thm 19.6: Suppose $f_\alpha : X \to Y_\alpha$. Define $f : X \to \Pi_{\alpha \in A} Y_\alpha$ by $f(x) = (f_\alpha(x))_{\alpha \in A}$. Let ΠX_α have the product topology. Then f is continuous is and only if f_α is continuous $\forall \; \alpha$