18. Continuous Functions

Defn: $f^{-1}(V) = \{x \mid f(x) \in V\}$.

Defn: $f : X \to Y$ is continuous iff for every V open in Y, $f^{-1}(V)$ is open in X.

Lemma: f continuous if and only if for every basis element B, $f^{-1}(B)$ is open in X.

Lemma: f continuous if and only if for every subbasis element S, $f^{-1}(S)$ is open in X.

Thm 18.1: Let $f : X \to Y$. Then the following are equivalent:

(1) f is continuous.

(2) For every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$.

(3) For every closed set B of Y, $f^{-1}(B)$ is closed in X.

(4) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

Defn: $f : X \to Y$ is a homeomorphism iff f is a bijection and both f and f^{-1} is continuous.

Defn: A property of a space X which is preserved by homeomorphisms is called a topological property of X.

Defn: $f : X \to Y$ is an imbedding of X in Y iff $f : X \to f(X)$ is a homeomorphism.

Thm 18.2

(a.) (Constant function) The constant map $f : X \to Y$, $f(x) = y_0$ is continuous.

(b.) (Inclusion) If A is a subspace of X, then the inclusion map $f : A \to X$, $f(a) = a$ is continuous.

(c.) (Composition) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

(d.) (Restricting the Domain) If $f : X \to Y$ is continuous and if A is a subspace of X, then the restricted function $f|_A : A \to Y$, $f|_A(a) = f(a)$ is continuous.

(e.) (Restricting or Expanding the Codomain) If $f : X \to Y$ is continuous and if Z is a subspace of Y containing the image set $f(X)$ or if Y is a subspace of Z, then $g : X \to Z$ is continuous.

(f.) (Local formulation of continuity) If $f : X \to Y$ and $X = \cup U_\alpha$, U_α open where $f|_{U_\alpha} U_\alpha \to Y$ is continuous, then $f : X \to Y$ is continuous.

Thm 18.3 (The pasting lemma): Let $X = A \cup B$ where A, B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, then $h : X \to Y$,

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases} \text{ is continuous.}$$

Thm 18.4: Let $f : A \to X \times Y$ be given by the equations $f(a) = (f_1(a), f_2(a))$ where $f_1 : A \to X$, $f_2 : A \to Y$. Then f is continuous if and only if f_1 and f_2 are continuous.

Defn: A *group* is a set, G, together with a function $* : G \times G \to G$, $*(a,b) = a * b$ such that

(0) Closure: $\forall a, b \in G$, $a * b \in G$.

(1) Associativity: $\forall a, b, c \in G$,
$$(a * b) * c = a * (b * c).$$

(2) Identity: $\exists\ e \in G$, such that $\forall a \in G$,
$$e * a = a * e = a.$$

(3) Inverses: $\forall a \in G$, $\exists a^{-1} \in G$ such that
$$a * a^{-1} = a^{-1} * a = e.$$

Defn: A group G is *commutative* or *abelian* if $\forall a, b \in G$, $a * b = b * a$.

Defn: A *topological group* is a set, G, such that

(1) G is a group.

(2) G is a topological space which is T_1.

(3) $* : G \times G \to G$, $*(a,b) = a * b$
and $i : G \to G$, $i(g) = g^{-1}$ are both continuous.