nth order LINEAR differential equation:

Thm 2.4.1: If p and g are continuous on (a, b) and the point $t_0 \in (a, b)$, then there exists a unique function $y = \phi(t)$ defined on (a, b) that satisfies the following initial value problem:

$$y' + p(t)y = g(t), \quad y(t_0) = y_0.$$

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),$$

$$y(t_0) = y_0, \quad y'(t_0) = y_1$$

Theorem 4.1.1: If $p_i : (a, b) \to R$, $i = 1, ..., n$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = g(t),$$

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \; ..., \quad y^{(n-1)}(t_0) = y_{n-1}$$

Proof: We proved the case $n = 1$ using an integrating factor. When $n > 1$, see more advanced textbook.

Example 4 from ch 2: $(t^2-1)y' + \frac{(t+1)y}{t-4} = ln|t|,\ y(3) = 6$

This equation is linear, so we know that it has a unique solution as long as p and g are continuous.

$$(t^2-1)y' + \frac{(t+1)y}{t-4} = ln|t| \ \Rightarrow\ 1y' + \frac{(t+1)}{(t-4)(t^2-1)}y = \frac{ln|t|}{t^2-1}$$

Note $p(t) = \frac{(t+1)}{(t-4)(t^2-1)} = \frac{(t+1)}{(t-4)(t+1)(t-1)} = \frac{1}{(t-4)(t-1)}$ is continuous for all $t \neq 1, 4$

Note $g(t) = \frac{ln|t|}{t^2-1} = \frac{ln|t|}{(t+1)(t-1)}$ is continuous for all $t \neq -1, 0, 1$

Thus $ty' - y = 1,\ y(t_0) = y_0$ has a unique solution as long as $t_0 \neq -1, 0, 1, 4$.

Since for IVP, $(t^2-1)y' + \frac{(t+1)y}{t-4} = ln|t|,\ y(3) = 6$, $t_0 = 3$, this IVP has a unique solution which by Thm 4.1.1 is valid on the interval $(1, 4)$.

NOTE: Theorem 4.1.1 is VERY useful in the real world. Suppose you can't solve the linear differential equation directly. You may be able to instead approximate the solution – see for example ch 5 series solution (guess $y = \sum a_n x^n$), which we won't cover in this class or MATH:3800 Elementary Numerical Analysis.

But your approximation is not of much use unless you know where your approximation is valid.

To solve $ay'' + by' + cy = g(t)$

1.) Easily solve homogeneous DE: $ay'' + by' + cy = 0$

$y = e^{rt} \Rightarrow ar^2 + br + c = 0 \Rightarrow y = c_1\phi_1 + c_2\phi_2$ for homogeneous solution (see sections 3.1, 3.3, 3.4, 4.1).

2.) More work: Find one solution to $ay'' + by' + cy = g(t)$ (see sections $3.5 = 4.3$, $\quad 3.6 = 4.4$)

If $y = \psi(t)$ is a soln to the nonhomogeneous DE, then general soln to $ay'' + by' + cy = g(t)$ is

$$y = c_1\phi_1 + c_2\phi_2 + \psi$$

Check:
$$a\phi_1'' + b\phi_1' + c\phi_1 = 0$$
$$a\phi_2'' + b\phi_2' + c\phi_2 = 0$$
$$a\psi'' + b\psi' + c\psi = g(t)$$

Note you can break step 2 into simpler parts. For example:

To solve $ay'' + by' + cy = g_1(t) + g_2(t)$

1.) Solve $ay'' + by' + cy = 0 \quad \Rightarrow \quad y = c_1\phi_1 + c_2\phi_2$ for homogeneous solution.

2a.) Solve $ay'' + by' + cy = g_1(t) \quad \Rightarrow \quad y = \psi_1$

2b.) Solve $ay'' + by' + cy = g_2(t) \quad \Rightarrow \quad y = \psi_2$

General solution to $ay'' + by' + cy = g_1(t) + g_2(t)$ is

$$y = c_1\phi_1 + c_2\phi_2 + \psi_1 + \psi_2$$

When does the following IVP have unique sol'n:

IVP: $ay'' + by' + cy = g(t)$, $y(t_0) = y_0$, $y'(t_0) = y_1$.

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t) + \psi(t)$ is a solution to $ay'' + by' + cy = g(t)$. Then $y' = c_1\phi_1'(t) + c_2\phi_2'(t) + \psi'(t)$

$y(t_0) = y_0$: $\quad y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0) + \psi(t_0)$

$y'(t_0) = y_1$: $\quad y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0) + \psi'(t_0)$

To find IVP solution, need to solve above system of two equations for the unknowns c_1 and c_2.

Note the IVP has a unique solution if and only if the above system of two equations has a unique solution for c_1 and c_2.

Note that in these equations c_1 and c_2 are the unknowns.

Let $b_0 = y_0 - \psi(t_0)$ and $b_1 = y_1 - \psi'(t_0)$

We can translate this linear system of equations into matrix form:

$$\begin{array}{l} c_1\phi_1(t_0) + c_2\phi_2(t_0) = b_0 \\ c_1\phi_1'(t_0) + c_2\phi_2'(t_0) = b_1 \end{array} \Rightarrow \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

Note this equation has a unique solution if and only if

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \phi_1\phi_2' - \phi_1'\phi_2 \neq 0$$

Definition: The Wronskian of two differential functions, ϕ_1 and ϕ_2 is

$$W(\phi_1, \phi_2) = \phi_1\phi_2' - \phi_1'\phi_2 = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$

Examples:

1.) $\text{W}(cos(t),\, sin(t)) = \begin{vmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{vmatrix}$

$$= cos^2(t) + sin^2(t) = 1 > 0.$$

2.) $\text{W}(e^{dt}cos(nt),\, e^{dt}sin(nt)) =$

$$\begin{vmatrix} e^{dt}cos(nt) & e^{dt}sin(nt) \\ de^{dt}cos(nt) - ne^{dt}sin(nt) & de^{dt}sin(nt) + ne^{dt}cos(nt) \end{vmatrix}$$

$$=e^{dt}cos(nt)(de^{dt}sin(nt)+ne^{dt}cos(nt))-e^{dt}sin(nt)(de^{dt}cos(nt)-ne^{dt}sin(n$$

$$=e^{2dt}[cos(nt)(dsin(nt)+ncos(nt))-sin(nt)(dcos(nt)-nsin(nt))]$$

$$=e^{2dt}[dcos(nt)sin(nt)+ncos^2(nt)-dsin(nt)cos(nt)+nsin^2(nt)])$$

$$= e^{2dt}[ncos^2(nt) + nsin^2(nt)]$$

$$= ne^{2dt}[cos^2(nt) + sin^2(nt)] \;=\; ne^{2dt} > 0 \text{ for all } t.$$

4.1: General Theory of nth Order Linear Eqns

When does the following IVP have a unique soln:

IVP: $y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = g(t)$,

$$y(t_0) = y_0,\ y'(t_0) = y_1, ..., y^{(n-1)}(t_0) = y_{n-1}.$$

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t) + ... + c_n\phi_n(t) + \psi(t)$ is the general solution to DE. Then

$y(t_0) = y_0$:

$$y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0) + ... + c_n\phi_n(t_0) + \psi(t_0)$$

$y'(t_0) = y_1$:

$$y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0) + ... + c_n\phi_n'(t_0) + \psi'(t_0)$$

$$\vdots$$

$y^{(n-1)}(t_0) = y_{n-1}$:

$$y_{n-1} = c_1\phi_1^{(n-1)}(t_0) + c_2\phi_2^{(n-1)}(t_0) + ... + c_n\phi_n^{(n-1)}(t_0) + \psi^{(n-1)}(t_0)$$

To find IVP solution, need to solve above system of equations for the unknowns c_i, $i = 1, ..., n$.

Note the IVP has a unique solution if and only if the above system of equations has a unique solution for the c_i's.

Let $b_k = y_k - \psi^{(k)}(t_0)$. Note that in these equations the c_i are the unknowns

Translating this linear system of eqns into matrix form:

$$\begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & ... & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & ... & \phi_n'(t_0) \\ & . & & \\ & . & & \\ & . & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & ... & \phi_n^{(n-1)}(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ . \\ . \\ . \\ b_{n-1} \end{bmatrix}$$

Note this equation has a unique solution if and only if

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & ... & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & ... & \phi_n'(t_0) \\ & . & & \\ & . & & \\ & . & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & ... & \phi_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

Defn: The Wronskian of the functions, ϕ_1, ϕ_2,..., ϕ_n is

$$W(\phi_1, \phi_2, ..., \phi_n) = det \begin{bmatrix} \phi_1(t) & \phi_2(t) & ... & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & ... & \phi_n'(t) \\ & . & & \\ & . & & \\ & . & & \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & ... & \phi_n^{(n-1)}(t) \end{bmatrix}$$

Note: $\{\phi_1, \phi_2, ..., \phi_n\}$ is a linearly independent set of fns if $W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$ for some t_0

In other words if ϕ_i are homogeneous solutions to an nth order linear DE,

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$$

and $W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$ for some t_0.

iff $\{\phi_1, \phi_2, ..., \phi_n\}$ is a basis for the solution set of this homogeneous equation.

In other words any homogeneous solution can be written as a linear combination of these basis elements:

$$y = c_1\phi_1 + ... + c_n\phi_n$$

Moreover, the general soln to the non-homogeneous eqn

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = g(t)$$

is just the translated version of the general homogeneous solution:

$$y = c_1\phi_1 + ... + c_n\phi_n + \psi$$

where ψ is a non-homogeneous solution.

Abel's theorem: if ϕ_i are homogeneous solutions to an nth order linear DE,

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$$

then $W(\phi_1, \phi_2, ..., \phi_n)(t) = ce^{-\int p_1(t)dt}$ for some constant c

$\phi_1, ..., \phi_n$ are linearly independent

iff

$c_1\phi_1(t) + ... + c_n\phi_n(t) = 0$ has a unique solution (that works for all t).

iff

the following system of equations has a unique solution

$$c_1\phi_1(t) + c_2\phi_2(t) + ... + c_n\phi_n(t) = 0$$
$$c_1\phi_1'(t) + c_2\phi_2'(t) + ... + c_n\phi_n'(t) = 0$$
$$\vdots$$
$$c_1\phi_1^{(n-1)}(t) + c_2\phi_2^{(n-1)}(t) + ... + c_n\phi_n^{(n-1)}(t) = 0$$

iff the following system of equations has a unique solution

$$\begin{bmatrix} \phi_1(t) & \phi_2(t) & ... & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & ... & \phi_n'(t) \\ & \vdots & & \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & ... & \phi_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Note this equation has a unique solution if and only if for some t_0

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & ... & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & ... & \phi_n'(t_0) \\ & . & & \\ & . & & \\ & . & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & ... & \phi_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

iff $\quad W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0,$

Example: Determine if $\{1 + 2t, 5 + 4t^2, 6 - 8t + 8t^2\}$ are linearly independent:

Method 1:
Solve $c_1(1 + 2t) + c_2(5 + 4t^2) + c_3(6 - 8t + 8t^2) = 0$

Or equivalently,

$$\text{solve } c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 6 \\ -8 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Or equivalently, solve $\begin{bmatrix} 1 & 5 & 6 \\ 2 & 0 & -8 \\ 0 & 4 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Method 2: Check the Wronskian

$$det \begin{bmatrix} 1 + 2t & 5 + 4t^2 & 6 - 8t + 8t^2 \\ 2 & 8t & -8 + 16t \\ 0 & 8 & 16 \end{bmatrix}$$