Summary of sections 3.1, 3, 4: Solve linear homogeneous 2nd order DE with constant coefficients.

Solve $ay'' + by' + cy = 0$. Educated guess $y = e^{rt}$, then

$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$

If $b^2 - 4ac > 0$, general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$.

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

general solution is $y = c_1e^{dt}cos(nt) + c_2e^{dt}sin(nt)$ where $r = d \pm in$

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: te^{r_1t}

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$.

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y'_0$ to solve for c_1, c_2 to find unique solution.

Examples:

Ex 1: Solve $y'' - 3y' - 4y = 0$, $y(0) = 1$, $y'(0) = 2$.

$$\text{If } y = e^{rt}, \text{ then } y' = re^{rt} \text{ and } y'' = r^2 e^{rt}.$$

$r^2 e^{rt} - 3re^{rt} - 4e^{rt} = 0$

$r^2 - 3r - 4 = 0$ implies $(r-4)(r+1) = 0$. Thus $r = 4, -1$

$$\text{Hence general solution is } y = c_1 e^{4t} + c_2 e^{-t}$$

Solution to IVP:
Need to solve for 2 unknowns, c_1 & c_2; thus need 2 eqns:

$y = c_1 e^{4t} + c_2 e^{-t}$, $\quad y(0) = 1$ implies $1 = c_1 + c_2$

$y' = 4c_1 e^{4t} - c_2 e^{-t}$, $\quad y'(0) = 2$ implies $2 = 4c_1 - c_2$

Thus $3 = 5c_1$ & hence $c_1 = \frac{3}{5}$ and $c_2 = 1 - c_1 = 1 - \frac{3}{5} = \frac{2}{5}$

$$\text{Thus IVP soln: } y = \tfrac{3}{5} e^{4t} + \tfrac{2}{5} e^{-t}$$

Ex 2: Solve $y'' - 3y' + 4y = 0$.

$y = e^{rt}$ implies $r^2 - 3r + 4 = 0$ and hence

$$r = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(4)}}{2} = \frac{3}{2} \pm \frac{\sqrt{9-16}}{2} = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

Hence general sol'n is $y = c_1 e^{\frac{3}{2}t} cos(\frac{\sqrt{7}}{2}t) + c_2 e^{\frac{3}{2}t} sin(\frac{\sqrt{7}}{2}t)$

Ex 3: $y'' - 6y' + 9y = 0$ implies $r^2 - 6r + 9 = (r-3)^2 = 0$
Repeated root, $r = 3$ implies

$$\text{general solution is } y = c_1 e^{3t} + c_2 t e^{3t}$$

So why did we guess $y = e^{rt}$?

Goal: Solve linear homogeneous 2nd order DE with constant coefficients,

$$ay'' + by' + cy = 0 \text{ where } a, b, c \text{ are constants}$$

Standard mathematical technique: make up simpler problems and see if you can generalize to the problem of interest.

Ex: linear homogeneous 1rst order DE: $y' + 2y = 0$

integrating factor $u(t) = e^{\int 2dt} = e^{2t}$

$y'e^{2t} + 2e^{2t}y = 0$

$(e^{2t}y)' = 0$. Thus $\int (e^{2t}y)'dt = \int 0dt$. Hence $e^{2t}y = C$

So $y = Ce^{-2t}$.

Thus exponential function could also be a solution to a linear homogeneous 2nd order DE

Ex: Simple linear homog 2nd order DE $y'' + 2y' = 0$.

Let $v = y'$, then $v' = y''$

$y'' + 2y' = 0$ implies $v' + 2v = 0$ implies $v = e^{2t}$.

Thus $v = y' = \frac{dy}{dt} = Ce^{-2t}$. Hence $dy = Ce^{-2t}dt$ and

$$y = c_1 e^{-2t} + c_2.$$

$$y = c_1 e^{-2t} + c_2.$$

Note 2 integrations give us 2 constants.

Note also that we the general solution is a linear combination of two solutions:

Let $c_1 = 1,\ c_2 = 0$, then we see, $y(t) = e^{-2t}$ is a solution.

Let $c_1 = 0,\ c_2 = 1$, then we see, $y(t) = 1$ is a solution.

The general solution is a linear combination of two solutions:

$$y = c_1 e^{-2t} + c_2(1).$$

Recall: you have seen this before:

Solve linear homogeneous matrix equation $A\mathbf{y} = \mathbf{0}$.

The general solution is a linear combination of linearly independent vectors that span the solution space:

$$\mathbf{y} = c_1\mathbf{v_1} + ...c_n\mathbf{v_n}$$

FYI: You could see this again:

Math 4050: Solve homogeneous linear recurrance relation $x_n - x_{n-1} - x_{n-2} = 0$ where $x_1 = 1$ and $x_2 = 1$.

Fibonacci sequence: $x_n = x_{n-1} + x_{n-2}$

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ ...$$

Note $x_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

Proof: $x_n = x_{n-1} + x_{n-2}$ implies $x_n - x_{n-1} - x_{n-2} = 0$

Suppose $x_n = r^n$. Then $x_{n-1} = r^{n-1}$ and $x_{n-2} = r^{n-2}$

Then $0 = x_n - x_{n-1} - x_{n-2} = r^n - r^{n-1} - r^{n-2}$

Thus $r^{n-2}(r^2 - r - 1) = 0$.

Thus either $r = 0$ or $r = \frac{1 \pm \sqrt{1-4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$

Thus $x_n = 0$, $\quad x_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$ and $\quad f_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$

are 3 different sequences that satisfy the

homog linear recurrence relation: $x_n - x_{n-1} - x_{n-2} = 0$.

Hence $x_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$ also satisfies this

homogeneous linear recurrence relation.

Suppose the initial conditions are $x_1 = 1$ and $x_2 = 1$

Then for $n = 1$: $x_1 = 1$ implies $c_1 + c_2 = 1$

For $n = 2$: $x_2 = 1$ implies $c_1 \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$

We can solve this for c_1 and c_2 to determine that

$$x_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Second order differential equation:

Linear equation with constant coefficients:
If the second order differential equation is

$$ay'' + by' + cy = 0,$$

$$\text{then } y = e^{rt} \text{ is a solution}$$

Need to have two independent solutions.

Solve the following IVPs:

1.) $y'' - 6y' + 9y = 0$ $\qquad y(0) = 1,\ y'(0) = 2$

2.) $4y'' - y' + 2y = 0$ $\qquad y(0) = 3,\ y'(0) = 4$

3.) $4y'' + 4y' + y = 0$ $\qquad y(0) = 6,\ y'(0) = 7$

4.) $2y'' - 2y = 0$ $\qquad y(0) = 5,\ y'(0) = 9$

Derivation of general solutions:

Ch 3: we guessed e^{rt} is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.1: If $b^2 - 4ac > 0$: $\quad y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$e^{it} = cos(t) + isin(t)$$

Hence $e^{(d+in)t} = e^{dt} e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$= c_1 e^{dt}[cos(nt) + isin(nt)] + c_2 e^{dt}[cos(-nt) + isin(-nt)]$

$= c_1 e^{dt} cos(nt) + ic_1 e^{dt} sin(nt) + c_2 e^{dt} cos(nt) - ic_2 e^{dt} sin(nt)$

$= (c_1 + c_2) e^{dt} cos(nt) + i(c_1 - c_2) e^{dt} sin(nt)$

$$= k_1 e^{dt} cos(nt) + k_2 e^{dt} sin(nt)$$

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1 t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $y = v(t)e^{rt}$?

$y' = v'(t)e^{rt} + v(t)re^{rt}$

$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}$

$\quad = v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}$

$ay'' + by' + cy = 0$

$a(v''e^{rt} + 2v're^{rt} + vr^2e^{rt}) + b(v'e^{rt} + vre^{rt}) + cve^{rt} = 0$

$a(v''(t) + 2v'(t)r + v(t)r^2) + b(v'(t) + v(t)r) + cv(t) = 0$

$av''(t) + 2av'(t)r + av(t)r^2 + bv'(t) + bv(t)r + cv(t) = 0$

$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$

$av''(t) + (2a(\frac{-b}{2a}) + b)v'(t) + 0 = 0$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$

$av''(t) + (-b + b)v'(t) = 0.$ Thus $av''(t) = 0.$

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1 t + k_2$

Hence $v(t)e^{r_1 t} = (k_1 t + k_2)e^{r_1 t}$ is a soln

Thus $te^{r_1 t}$ is a nice second solution.

Hence general solution is $y = c_1 e^{r_1 t} + c_2 te^{r_1 t}$