Existence and Uniqueness for LINEAR DEs.

Homogeneous:

$$y^{(n)} + p_1(t)y^{(n-1)} + ...p_{n-1}(t)y' + p_n(t)y = 0$$

Non-homogeneous: $g(t) \neq 0$

$$y^{(n)} + p_1(t)y^{(n-1)} + ...p_{n-1}(t)y' + p_n(t)y = g(t)$$

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the

$$\text{IVP: } y' + p(t)y = g(t), \quad y(t_0) = y_0$$

Thm: If $y = \phi_1(t)$ is a solution to homogeneous equation, $y' + p(t)y = 0$, then $y = c\phi_1(t)$ is the general solution to this equation.

If in addition $y = \psi(t)$ is a solution to non-homogeneous equation, $y' + p(t)y = g(t)$, then $y = c\phi_1(t) + \psi(t)$ is the general solution to this equation.

Partial proof: $y = \phi_1(t)$ is a solution to $y' + p(t)y = 0$ implies

Thus $y = c\phi_1(t)$ is a solution to $y' + p(t)y = 0$ since

$y = \psi(t)$ is a solution to $y' + p(t)y = g(t)$ implies

Thus $y = c\phi_1(t) + \psi(t)$ is a solution to $y' + p(t)y = g(t)$ since

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),$$
$$y(t_0) = y_0,$$
$$y'(t_0) = y_0'$$

Thm 3.2.2: If ϕ_1 and ϕ_2 are two solutions to a homogeneous linear differential equation, then $c_1\phi_1 + c_2\phi_2$ is also a solution to this linear differential equation.

Proof of thm 3.2.2:

Since $y(t) = \phi_i(t)$ is a solution to the linear homogeneous differential equation $y'' + py' + qy = 0$ where p and q are functions of t (note this includes the case with constant coefficients), then

Claim: $y(t) = c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to $y'' + py' + qy = 0$

Pf of claim:

Second order differential equation:

Linear equation with constant coefficients:
If the second order differential equation is

$$ay'' + by' + cy = 0,$$

$$\text{then } y = e^{rt} \text{ is a solution}$$

Need to have two independent solutions.

Solve the following IVPs:

1.) $y'' - 6y' + 9y = 0$ $\qquad y(0) = 1,\ y'(0) = 2$

2.) $4y'' - y' + 2y = 0$ $\qquad y(0) = 3,\ y'(0) = 4$

3.) $4y'' + 4y' + y = 0$ $\qquad y(0) = 6,\ y'(0) = 7$

4.) $2y'' - 2y = 0$ $\qquad y(0) = 5,\ y'(0) = 9$

$ay'' + by' + cy = 0, \quad y = e^{rt}$, then
$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$

If $b^2 - 4ac > 0$, general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$.

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

general solution is $y = c_1e^{dt}cos(nt) + c_2e^{dt}sin(nt)$ where $r = d \pm in$

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: te^{r_1t}

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$.

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y_0'$ to solve for c_1, c_2 to find unique solution.

Derivation of general solutions:

If $b^2 - 4ac > 0$ we guessed e^{rt} is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$e^{it} = cos(t) + isin(t)$$

Hence $e^{(d+in)t} = e^{dt} e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$= c_1 e^{dt}[cos(nt) + isin(nt)] + c_2 e^{dt}[cos(-nt) + isin(-nt)]$

$= c_1 e^{dt} cos(nt) + ic_1 e^{dt} sin(nt) + c_2 e^{dt} cos(nt) - ic_2 e^{dt} sin(nt)$

$=(c_1 + c_2) e^{dt} cos(nt) + i(c_1 - c_2) e^{dt} sin(nt)$

$$= k_1 e^{dt} cos(nt) + k_2 e^{dt} sin(nt)$$

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1 t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $y = v(t)e^{rt}$?

$$y' = v'(t)e^{rt} + v(t)re^{rt}$$
$$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}$$
$$= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}$$

$$ay'' + by' + cy = 0$$

$$a(v''e^{rt} + 2v're^{rt} + vr^2e^{rt}) + b(v'e^{rt} + vre^{rt}) + cve^{rt} = 0$$

$$a(v''(t) + 2v'(t)r + v(t)r^2) + b(v'(t) + v(t)r) + cv(t) = 0$$

$$av''(t) + 2av'(t)r + av(t)r^2 + bv'(t) + bv(t)r + cv(t) = 0$$

$$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$$

$$av''(t) + (2a(\tfrac{-b}{2a}) + b)v'(t) + 0 = 0$$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$

$av''(t) + (-b + b)v'(t) = 0.$ Thus $av''(t) = 0.$

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1 t + k_2$

Hence $v(t)e^{r_1 t} = (k_1 t + k_2)e^{r_1 t}$ is a soln

Thus $te^{r_1 t}$ is a nice second solution.

Hence general solution is $y = c_1 e^{r_1 t} + c_2 te^{r_1 t}$

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

Since $r = 0 \pm 1i$, $y = k_1 cos(t) + k_2 sin(t)$.

Then $y' = -k_1 sin(t) + k_2 cos(t)$

$y(0) = -1$: $-1 = k_1 cos(0) + k_2 sin(0)$ implies $-1 = k_1$

$y'(0) = -3$: $-3 = -k_1 sin(0) + k_2 cos(0)$ implies $-3 = k_2$

Thus IVP solution: $y = -cos(t) - 3sin(t)$

When does the following IVP have unique sol'n:

IVP: $ay'' + by' + cy = 0$, $y(t_0) = y_0$, $y'(t_0) = y_1$.

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t)$ is a solution to
$ay'' + by' + cy = 0$. Then $y' = c_1\phi_1'(t) + c_2\phi_2'(t)$

$y(t_0) = y_0$: $y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0)$

$y'(t_0) = y_1$: $y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0)$

To find IVP solution, need to solve above system of two equations for the unknowns c_1 and c_2.

Note the IVP has a unique solution if and only if the above system of two equations has a unique solution for c_1 and c_2.

Note that in these equations c_1 and c_2 are the unknowns and $y_0, \phi_1(t_0), \phi_2(t_0)$, $y_1, \phi_1'(t_0), \phi_2'(t_0)$ are the constants. We can translate this linear system of equations into matrix form:

$$\begin{array}{l} c_1\phi_1(t_0) + c_2\phi_2(t_0) = y_0 \\ c_1\phi_1'(t_0) + c_2\phi_2'(t_0) = y_1 \end{array} \Rightarrow \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

Note this equation has a unique solution if and only if

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \phi_1\phi_2' - \phi_1'\phi_2 \neq 0$$

Definition: The Wronskian of two differential functions, ϕ_1 and ϕ_2 is

$$W(\phi_1, \phi_2) = \phi_1\phi_2' - \phi_1'\phi_2 = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$

Examples:

1.) $\text{W}(cos(t),\, sin(t)) = \begin{vmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{vmatrix}$

$$= cos^2(t) + sin^2(t) = 1 > 0.$$

2.) $\text{W}(e^{dt}cos(nt),\, e^{dt}sin(nt)) =$

$$\begin{vmatrix} e^{dt}cos(nt) & e^{dt}sin(nt) \\ de^{dt}cos(nt) - ne^{dt}sin(nt) & de^{dt}sin(nt) + ne^{dt}cos(nt) \end{vmatrix}$$

$$=e^{dt}cos(nt)(de^{dt}sin(nt)+ne^{dt}cos(nt))-e^{dt}sin(nt)(de^{dt}cos(nt)-ne^{dt}sin(nt))$$

$$=e^{2dt}[cos(nt)(dsin(nt)+ncos(nt))-sin(nt)(dcos(nt)-nsin(nt))]$$

$$=e^{2dt}[dcos(nt)sin(nt)+ncos^2(nt)-dsin(nt)cos(nt)+nsin^2(nt)])$$

$$= e^{2dt}[ncos^2(nt) + nsin^2(nt)]$$

$$= ne^{2dt}[cos^2(nt) + sin^2(nt)] \ = \ ne^{2dt} > 0 \text{ for all } t.$$

Definition: The Wronskian of two differential functions, f and g is

$$W(f,g) = fg' - f'g = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

Thm 3.2.3: Suppose that
ϕ_1 and ϕ_2 are two solutions to $y''+p(t)y'+q(t)y = 0$.
If $W(\phi_1, \phi_2)(t_0) = \phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0) \neq 0$, then
there is a unique choice of constants c_1 and c_2 such that $c_1\phi_1+c_2\phi_2$ satisfies this homogeneous linear differential equation and initial conditions, $y(t_0) = y_0$, $y'(t_0) = y_0'$.

Thm 3.2.4: Given the hypothesis of thm 3.2.1, suppose that ϕ_1 and ϕ_2 are two solutions to

$$y'' + p(t)y' + q(t)y = 0.$$

If $W(\phi_1, \phi_2)(t_0) \neq 0$, for some $t_0 \in (a, b)$, then any solution to this homogeneous linear differential equation can be written as $y = c_1\phi_1 + c_2\phi_2$ for some c_1 and c_2.

Defn If ϕ_1 and ϕ_2 satisfy the conditions in thm 3.2.4, then ϕ_1 and ϕ_2 form a fundamental set of solutions to $y'' + p(t)y' + q(t)y = 0$.

Thm 3.2.5: Given any second order homogeneous linear differential equation, there exist a pair of functions which form a fundamental set of solutions.

3.3: Linear Independence and the Wronskian

Defn: f and g are linearly dependent if there exists constants c_1, c_2 such that $c_1 \neq 0$ or $c_2 \neq 0$ and

$$c_1 f(t) + c_2 g(t) = 0 \text{ for all } t \in (a, b)$$

Thm 3.3.1: If $f : (a, b) \to R$ and $g(a, b) \to R$ are differentiable functions on (a, b) and if $W(f, g)(t_0) \neq 0$ for some $t_0 \in (a, b)$, then f and g are linearly independent on (a, b). Moreover, if f and g are linearly dependent on (a, b), then $W(f, g)(t) = 0$ for all $t \in (a, b)$

If $c_1 f(t) + c_2 g(t) = 0$ for all t, then $c_1 f'(t) + c_2 g'(t) = 0$

Solve the following linear system of equations for c_1, c_2

$$c_1 f(t_0) + c_2 g(t_0) = 0$$
$$c_1 f'(t_0) + c_2 g'(t_0) = 0$$

$$\begin{bmatrix} f(t_0) & g(t_0) \\ f'(t_0) & g'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thm: Suppose $c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to

$$ay'' + by' + cy = 0,$$

If ψ is a solution to

$$ay'' + by' + cy = g(t)\ [*],$$

Then $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to [*].

Moreover if γ is also a solution to [*], then there exist constants c_1, c_2 such that

$$\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$$

Or in other words, $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to [*].

Proof:

Define $L(f) = af'' + bf' + cf$.

Recall L is a linear function.

Let $h = c_1\phi_1(t) + c_2\phi_2(t)$. Since h is a solution to the differential equation, $ay'' + by' + cy = 0$,

Since ψ is a solution to $ay'' + by' + cy = g(t)$, We will now show that $\psi + c_1\phi_1(t) + c_2\phi_2(t) = \psi + h$ is also a solution to [*].

Since γ a solution to $ay'' + by' + cy = g(t)$,

We will first show that $\gamma - \psi$ is a solution to the differential equation $ay'' + by' + cy = 0$.

Since $\gamma - \psi$ is a solution to $ay'' + by' + cy = 0$ and

$c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to

$$ay'' + by' + cy = 0,$$

there exist constants c_1, c_2 such that

$$\gamma - \psi = \underline{\hspace{8cm}}$$

Thus $\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$.

Thm:
Suppose f_1 is a a solution to $ay'' + by' + cy = g_1(t)$ and f_2 is a a solution to $ay'' + by' + cy = g_2(t)$, then $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$

Proof: Let $L(f) = af'' + bf' + cf$.

Since f_1 is a solution to $ay'' + by' + cy = g_1(t)$,

Since f_2 is a solution to $ay'' + by' + cy = g_2(t)$,

We will now show that $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$.

Sidenote: The proofs above work even if a, b, c are functions of t instead of constants.

Examples: Find a suitable form for ψ for the following differential equations:

1.) $y'' - 4y' - 5y = 4e^{2t}$

2.) $y'' - 4y' - 5y = t^2 - 2t + 1$

3.) $y'' - 4y' - 5y = 4sin(3t)$

4.) $y'' - 5y = 4sin(3t)$

5.) $y'' - 4y' = t^2 - 2t + 1$

6.) $y'' - 4y' - 5y = 4(t^2 - 2t - 1)e^{2t}$

7.) $y'' - 4y' - 5y = 4sin(3t)e^{2t}$

8.) $y'' - 4y' - 5y = 4(t^2 - 2t - 1)sin(3t)e^{2t}$

9.) $y'' - 4y' - 5y = 4sin(3t) + 4sin(3t)e^{2t}$

10.) $y'' - 4y' - 5y$
$= 4sin(3t)e^{2t} + 4(t^2 - 2t - 1)e^{2t} + t^2 - 2t - 1$

11.) $y'' - 4y' - 5y = 4sin(3t) + 5cos(3t)$

12.) $y'' - 4y' - 5y = 4e^{-t}$

To solve $ay'' + by' + cy = g_1(t) + g_2(t) + ...g_n(t)$ [**]

1.) Find the general solution to $ay'' + by' + cy = 0$:

$$c_1\phi_1 + c_2\phi_2$$

2.) For each g_i, find a solution to $ay'' + by' + cy = g_i$:

$$\psi_i$$

This includes plugging guessed solution ψ_i into $ay'' + by' + cy = g_i$.

The general solution to [**] is

$$c_1\phi_1 + c_2\phi_2 + \psi_1 + \psi_2 + ...\psi_n$$

3.) If initial value problem:

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2).

Solve $y'' - 4y' - 5y = 4sin(3t)$, $y(0) = 6$, $y'(0) = 7$.

1.) **First solve homogeneous equation:**

Find the general solution to $y'' - 4y' - 5y = 0$:

Guess $y = e^{rt}$ for HOMOGENEOUS equation:

$y' = re^{rt}$, $y' = r^2e^{rt}$

$y'' - 4y' - 5y = 0$

$r^2e^{rt} - 4re^{rt} - 5e^{rt} = 0$

$e^{rt}(r^2 - 4r - 5) = 0$

$e^{rt} \neq 0$, thus can divide both sides by e^{rt}:

$$r^2 - 4r - 5 = 0$$

$(r + 1)(r - 5) = 0$. Thus $r = -1, 5$.

Thus $y = e^{-t}$ and $y = e^{5t}$ are both solutions to LINEAR HOMOGENEOUS equation.

Thus the general solution to the 2nd order LINEAR HOMOGENEOUS equation is

$$y = c_1e^{-t} + c_2e^{5t}$$

2.) **Find one solution to non-homogeneous eq'n:**

Find a solution to $ay'' + by' + cy = 4sin(3t)$:

Guess $y = Asin(3t) + Bcos(3t)$

$$y' = 3Acos(3t) - 3Bsin(3t)$$

$$y'' = -9Asin(3t) - 9Bcos(3t)$$

$y'' - 4y' - 5y = 4sin(3t)$

$$\begin{array}{ccccc} -9Asin(3t) & - & 9Bcos(3t) & & \\ 12Bsin(3t) & - & 12Acos(3t) & & \\ -5Asin(3t) & - & 5cos(3t) & & \\ \hline (12B - 14A)sin(3t) & - & (-14B - 12A)cos(3t) & = & 4sin(3t) \end{array}$$

Since $\{sin(3t), cos(3t)\}$ is a linearly independent set:

$12B - 14A = 4$ and $-14B - 12A = 0$

Thus $A = -\frac{14}{12}B = -\frac{7}{6}B$ and

$12B - 14(-\frac{7}{6}B) = 12B + 7(\frac{7}{3}B) = \frac{36+49}{3}B = \frac{85}{3}B = 4$

Thus $B = 4(\frac{3}{85}) = \frac{12}{85}$ and $A = -\frac{7}{6}B = -\frac{7}{6}(\frac{12}{85}) = -\frac{14}{85}$

Thus $y = (-\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$ is one solution to the nonhomogeneous equation.

Thus the general solution to the 2nd order linear non-homogeneous equation is

$$y = c_1e^{-t} + c_2e^{5t} - (\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$$

3.) **If initial value problem:**

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2).

NOTE: you must know the GENERAL solution to the ODE BEFORE you can solve for the initial values. The homogeneous solution and the one nonhomogeneous solution found in steps 1 and 2 above do NOT need to separately satisfy the initial values.

Solve $y'' - 4y' - 5y = 4sin(3t), \;\; y(0) = 6, \, y'(0) = 7.$

General solution: $y = c_1e^{-t} + c_2e^{5t} - (\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$

Thus $y' = -c_1e^{-t} + 5c_2e^{5t} - (\frac{42}{85})cos(3t) - \frac{36}{85}sin(3t)$

$y(0) = 6$: $\quad 6 = c_1 + c_2 + \frac{12}{85} \quad\quad \frac{498}{85} = c_1 + c_2$

$y'(0) = 7$: $\quad 7 = -c_1 + 5c_2 - \frac{42}{85} \quad\quad \frac{637}{85} = -c_1 + 5c_2$

$6c_2 = \frac{498+637}{85} = \frac{1135}{85} = \frac{227}{17}$. Thus $c_2 = \frac{227}{102}$.

$c_1 = \frac{498}{85} - c_2 = \frac{498}{85} - \frac{227}{102} = \frac{2988-1135}{510} = \frac{1853}{510} = \frac{109}{30}$

Thus $y = (\frac{109}{30})e^{-t} + (\frac{227}{102})e^{5t} - (\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$.

Partial Check: $y(0) = (\frac{109}{30}) + (\frac{227}{102}) + \frac{12}{85} = 6.$

$$y'(0) = -\frac{109}{30} + 5(\frac{227}{102}) - \frac{42}{85} = 7.$$

$(e^{-t})'' - 4(e^{-t})' - 5(e^{-t}) = 0$ and $(e^{5t})'' - 4(e^{5t})' - 5(e^{5t}) = 0$

Potential proofs for exam 1:

Proof by (counter) example:

1. Prove a function is not 1:1, not onto, not a bijection, not linear.

2. Prove that a differential equation can have multiple solutions.

Prove convergence of a series using ratio test.

Induction proof.

Prove a function is linear.

Theorem 3.2.2: If $y = \phi_1(t)$ and $y = \phi_2(t)$ are solutions to the 2nd order linear ODE, $ay'' + by' + cy = 0$, then their linear combination $y = c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution for constants c_1 and c_2.

Note you may use what you know from both pre-calculus and calculus (e.g., integration and derivatives are linear).