Calulus pre-requisites you must know.

Derivative = slope of tangent line = rate.

Integral = area between curve and x-axis (where area can be negative).

The Fundamental Theorem of Calculus: Suppose f continuous on $[a, b]$.

1.) If $G(x) = \int_a^x f(t)dt$, then $G'(x) = f(x)$.

$$\text{I.e., } \frac{d}{dx}[\int_a^x f(t)dt] = f(x).$$

2.) $\int_a^b f(t)dt = F(b) - F(a)$ where F is any antiderivative of f, that is $F' = f$.

Integration Pre-requisites:

* Integration by substitution

* Integration by parts

* Integration by partial fractions

Note: Partial fractions are also used in ch 6 for a different application.

Suppose f is cont. on (a, b) and the point $t_0 \in (a, b)$,

Solve IVP: $\frac{dy}{dt} = f(t), \qquad y(t_0) = y_0$

$$dy = f(t)dt$$

$$\int dy = \int f(t)dt$$

$y = F(t) + C$ where F is any anti-derivative of F.

Initial Value Problem (IVP): $y(t_0) = y_0$

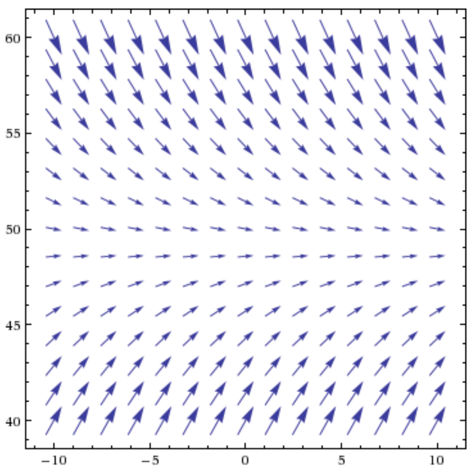
$$y_0 = F(t_0) + C \text{ implies } C = y_0 - F(t_0)$$

Hence unique solution (if domain connected) to IVP:

$$y = F(t) + y_0 - F(t_0)$$

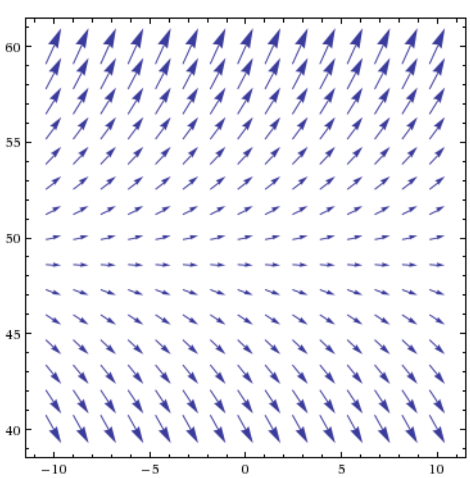
1.1: Examples of differentiable equation:

1.) $F = ma = m\frac{dv}{dt} = mg - \gamma v$



2.) Mouse population increases at a rate proportional to the current population:

More general model : $\frac{dp}{dt} = rp - k$
where $p(t)$ = mouse population at time t,
r = growth rate or rate constant,
k = predation rate = # mice killed per unit time.



3.) Continuous compounding $\frac{dS}{dt} = rS + k$
where $S(t)$ = amount of money at time t,
r = interest rate,
k = constant deposit rate

---

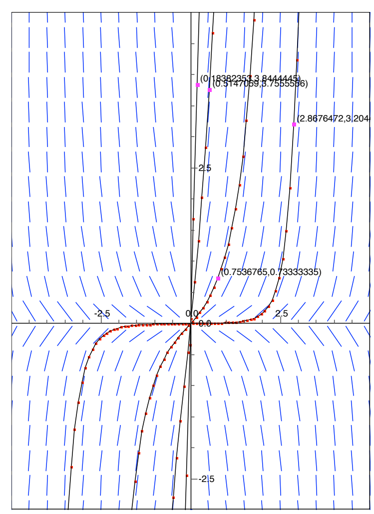
direction field = slope field = graph of $\frac{dy}{dt}$ in t, y-plane.

*** can use slope field to determine behavior of y including as $t \to \pm\infty$.

*** Equilibrium Solution = constant solution

Most differential equations do not have an equilibrium solution.

Initial value: A chosen point (t_0, y_0) through which a solution must pass. I.e., (t_0, y_0) lies on the graph of the solution that satisfies this initial value.



Initial value problem (IVP): A differential equation where initial value is specified.

An initial value problem can have 0, 1, or multiple equilibrium solutions.

************Existence of a solution************

************Uniqueness of solution************

1.3:

ODE (ordinary differential equation): single independent variable

Ex: $\frac{dy}{dt} = ay + b$

PDE (partial differential equation): several independent variables

Ex: $\frac{\partial xy}{\partial x} = \frac{\partial xy}{\partial y}$

---

order of differential eq'n: order of highest derivative

example of order n: $y^{(n)} = f(t, y, ..., y^{(n-1)})$

Linear vs Non-linear

Linear: $a_0 y^{(n)} + ... + a_{n-1} y' + a_n y = g(t)$

where a_i's are functions of t

Note for this linear equation, the left hand side is a linear combination of the derivatives of y (denoted by $y^{(k)}, k = 0, ..., n$) where the coefficient of $y^{(k)}$ is a function of t (denoted $a_k(t)$).

Linear: $a_0(t)y^{(n)} + ... + a_{n-1}(t)y' + a_n(t)y = g(t)$

Determine if linear or non-linear:

Ex: $ty'' - t^3y' - 3y = sin(t)$

Ex: $2y'' - 3y' - 3y^2 = 0$

Show that for some value of r, $y = e^{rt}$ is a soln to the 1^{rst} order linear homogeneous equation $2y' - 6y = 0$.

To show something is a solution, plug it in:

$y = e^{rt}$ implies $y' = re^{rt}$. Plug into $2y' - 6y = 0$:

$2re^{rt} + 6e^{rt} = 0$ implies $2r - 6 = 0$ implies $r = 3$

Thus $y = e^{3t}$ is a solution to $2y' - 6y = 0$.

Show $y = Ce^{3t}$ is a solution to $2y' - 6y = 0$.

$$2y' - 6y = 2(Ce^{3t})' - 6(Ce^{3t}) = 2C(e^{3t})' - 6C(e^{3t}) \\ = C[2(e^{3t})' - 6(e^{3t})] = C(0) = 0.$$

If $y(0) = 4$, then $4 = Ce^{3(0)}$ implies $C = 4$.

Thus by existence and uniqueness thm, $y = 4e^{3t}$ is the unique solution to IVP: $2y' + 6y = 0$, $y(0) = 4$.

CH 2: Solve $\frac{dy}{dt} = f(t, y)$ for special cases:

2.2: Separation of variables: $N(y)dy = P(t)dt$

2.1: First order linear eqn: $\frac{dy}{dt} + p(t)y = g(t)$

Ex 1: $t^2y' + 2ty = tsin(t)$

Ex 2: $y' = ay + b$

Ex 3: $y' + 3t^2y = t^2$, $y(0) = 0$

Note: can use either section 2.1 method (integrating factor) or 2.2 method (separation of variables) to solve ex 2 and 3.

Ex 1: $t^2y' + 2ty = sin(t)$
(note, cannot use separation of variables).

$t^2y' + 2ty = sin(t)$

$$(t^2y)' = sin(t) \quad \text{implies} \quad \int (t^2y)'dt = \int sin(t)dt$$

$(t^2y) = -cos(t) + C$ implies $y = -t^{-2}cos(t) + Ct^{-2}$

Ex. 2: Solve $\frac{dy}{dt} = ay + b$ by separating variables:

$$\frac{dy}{ay+b} = dt \quad \Rightarrow \quad \int \frac{dy}{ay+b} = \int dt \quad \Rightarrow \quad \frac{ln|ay+b|}{a} = t + C$$

$ln|ay + b| = at + C$ implies $e^{ln|ay+b|} = e^{at+C}$

$|ay + b| = e^C e^{at}$ implies $ay + b = \pm(e^C e^{at})$

$ay = Ce^{at} - b$ implies $y = Ce^{at} - \frac{b}{a}$

Gen ex: Solve $y' + p(x)y = g(x)$

Let $F(x)$ be an anti-derivative of $p(x)$. Thus $p(x) = F'(x)$

$$e^{F(x)}y' + [p(x)e^{F(x)}]y = g(x)e^{F(x)}$$

$$e^{F(x)}y' + [F'(x)e^{F(x)}]y = g(x)e^{F(x)}$$

$$[e^{F(x)}y]' = g(x)e^{F(x)}$$

$$e^{F(x)}y = \int g(x)e^{F(x)}dx$$

$$y = e^{-F(x)} \int g(x)e^{F(x)}dx$$

2.3: Modeling with differential equations.

Suppose salty water enters and leaves a tank at a rate of 2 liters/minute.

Suppose also that the salt concentration of the water entering the tank varies with respect to time according to $Q(t) \cdot tsin(t^2)$ g/liters where $Q(t)$ = amount of salt in tank in grams. (Note: this is not realistic).

If the tank contains 4 liters of water and initially contains 5g of salt, find a formula for the amount of salt in the tank after t minutes.

Let $Q(t)$ = amount of salt in tank in grams.

Note $Q(0) = 5$ g

rate in = (2 liters/min)($Q(t) \cdot tsin(t^2)$ g/liters)

$$= 2Qtsin(t^2) \text{ g/min}$$

rate out = (2 liters/min)$(\frac{Q(t)g}{4\text{liters}}) = \frac{Q}{2}$ g/min

$\frac{dQ}{dt}$ = rate in - rate out = $2Qtsin(t^2) - \frac{Q}{2}$

$$\frac{dQ}{dt} = Q(2tsin(t^2) - \frac{1}{2}), \qquad Q(0) = 5$$

This is a first order linear ODE. It is also a separable ODE. Thus can use either 2.1 or 2.2 methods.

Using the easier 2.2:

$\int \frac{dQ}{Q} = \int (2tsin(t^2) - \frac{1}{2})dt = \int 2tsin(t^2)dt - \int \frac{1}{2}dt$

$$\text{Let } u = t^2,\ du = 2tdt$$

$$ln|Q| = \int sin(u)du - \frac{t}{2} = -cos(u) - \frac{t}{2} + C$$
$$= -cos(t^2) - \frac{t}{2} + C$$

$|Q| = e^{-cos(t^2) - \frac{t}{2} + C} = e^C e^{-cos(t^2) - \frac{t}{2}}$

$$Q = Ce^{-cos(t^2) - \frac{t}{2}}$$

$Q(0) = 5: \quad 5 = Ce^{-1-0} = Ce^{-1}$. Thus $C = 5e$

Thus $Q(t) = 5e \cdot e^{-cos(t^2) - \frac{t}{2}}$

$$\text{Thus } Q(t) = 5e^{-cos(t^2) - \frac{t}{2} + 1}$$

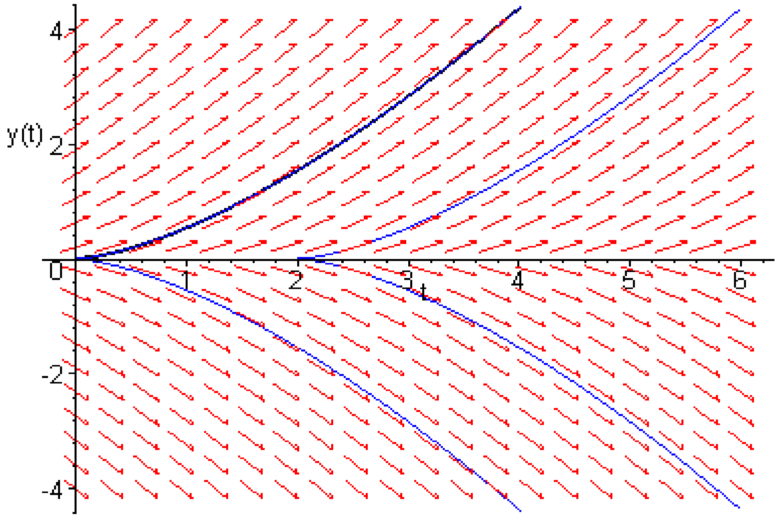
Long-term behaviour:

$$Q(t) = 5(e^{-cos(t^2)})(e^{\frac{-t}{2}})e$$

As $t \to \infty$, $e^{\frac{-t}{2}} \to 0$, while $5(e^{-cos(t^2)})e$ are finite.

Thus as $t \to \infty$, $Q(t) \to 0$.

Section 2.4 $y' = y^{1/3}$



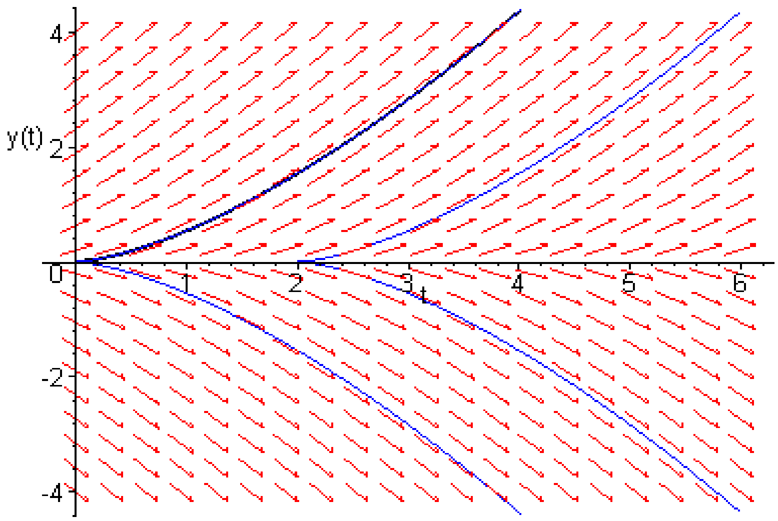


Figure 2.4.1 from ***Elementary Differential Equations and Boundary Value Problems*****, Eighth Edition** by William E. Boyce and Richard C. DiPrima

Note IVP, $y' = y^{\frac{1}{3}}$, $y(x_0) = 0$ has an infinite number of solutions,

while IVP, $y' = y^{\frac{1}{3}}$, $y(x_0) = y_0$ where $y_0 \neq 0$ has a unique solution.

Initial Value Problem: $y(t_0) = y_0$
Use initial value to solve for C.

.

Section 2.4: Existence and Uniqueness.

In general, for $y' = f(t, y)$, $y(t_0) = y_0$, solution may or may not exist and solution may or may not be unique.

Example Non-unique: $y' = y^{\frac{1}{3}}$

$y = 0$ is a solution to $y' = y^{\frac{1}{3}}$ since $y' = 0 = 0^{\frac{1}{3}} = y^{\frac{1}{3}}$

Suppose $y \neq 0$. Then $\frac{dy}{dx} = y^{\frac{1}{3}}$ implies $y^{-\frac{1}{3}} dy = dx$

$\int y^{-\frac{1}{3}} dy = \int dx$ implies $\frac{3}{2} y^{\frac{2}{3}} = x + C$

$y^{\frac{2}{3}} = \frac{2}{3}x + C$ implies $y = \pm\sqrt{(\frac{2}{3}x + C)^3}$

Suppose $y(3) = 0$. Then $0 = \sqrt{(2 + C)^3} \Rightarrow C = -2.$ $\blacksquare$

The IVP, $y' = y^{\frac{1}{3}}$, $y(3) = 0$, has an infinite # of sol'ns $\blacksquare$

including: $y = 0, \quad y = \sqrt{(\frac{2}{3}x - 2)^3}, \quad y = -\sqrt{(\frac{2}{3}x - 2)^3}$

Examples: No solution:

Ex 1: $y' = y' + 1$

Ex 2: $(y')^2 = -1$

Ex 3 (IVP): $\frac{dy}{dx} = y(1 + \frac{1}{x})$, $y(0) = 1$

$\int \frac{dy}{y} = \int (1 + \frac{1}{x})dx$ implies $ln|y| = x + ln|x| + C$

$|y| = e^{x+ln|x|+C} = e^x e^{ln|x|} e^C = C|x|e^x = Cxe^x$

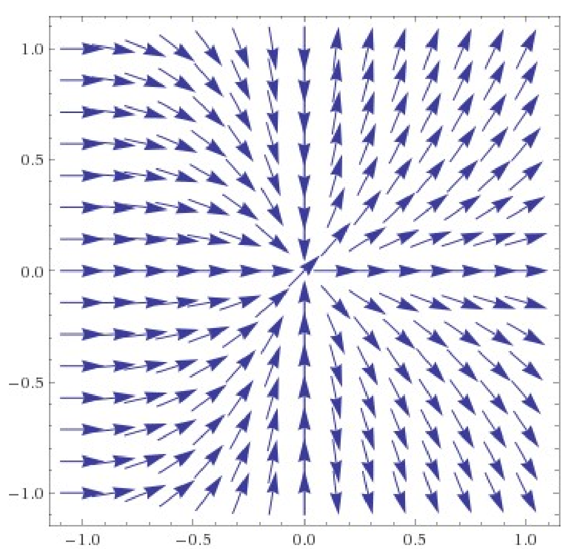
$y = \pm Cxe^x$ implies $y = Cxe^x$

$y(0) = 1$: $1 = C(0)e^0 = 0$ implies

IVP $\frac{dy}{dx} = y(1 + \frac{1}{x})$, $y(0) = 1$ has no solution.

http://www.wolframalpha.com

slope field: $\{1, y(1+1/x)\}/sqrt(1+y\wedge 2(1+1/x)\wedge 2)$



Special cases:

Suppose f is cont. on (a, b) and the point $t_0 \in (a, b)$, Solve IVP: $\frac{dy}{dt} = f(t)$, $y(t_0) = y_0$

$$dy = f(t)dt$$

$$\int dy = \int f(t)dt$$

$y = F(t) + C$ where F is any anti-derivative of F.

Initial Value Problem (IVP): $y(t_0) = y_0$

$$y_0 = F(t_0) + C \text{ implies } C = y_0 - F(t_0)$$

Hence unique solution (if domain connected) to IVP:

$$y = F(t) + y_0 - F(t_0)$$

First order linear differential equation:

Thm 2.4.1: If p and g are continuous on (a, b) and the point $t_0 \in (a, b)$, then there exists a unique function $y = \phi(t)$ defined on (a, b) that satisfies the following initial value problem:

$$y' + p(t)y = g(t), \quad y(t_0) = y_0.$$

More general case (but still need hypothesis)

Thm 2.4.2: Suppose the functions $z = f(t, y)$ and $z = \frac{\partial f}{\partial y}(t, y)$ are continuous on $(a, b) \times (c, d)$ and the point $(t_0, y_0) \in (a, b) \times (c, d)$,

then there exists an interval $(t_0 - h, t_0 + h) \subset (a, b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0.$$

If possible **without solving**, determine where the solution exists for the following initial value problems:

If not possible **without solving**, state where in the ty-plane, the hypothesis of theorem 2.4.2 is satisfied. In other words, use theorm 2.4.2 to determine where for some interval about t_0, a solution to IVP, $y' = f(t, y)$, $y(t_0) = y_0$ exists and is unique.

Example 1: $ty' - y = 1, \; y(t_0) = y_0$

Example 2: $y' = ln|\frac{t}{y}, \ y(3) = 6$

Example 3: $(t^2 - 1)y' - \frac{t^3 y}{t-4} = ln|t|, \ y(3) = 6$

Section 2.4 example: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$F(y,t) = \frac{1}{(1-t)(2-y)}$ is continuous for all $t \neq 1$, $y \neq 2$

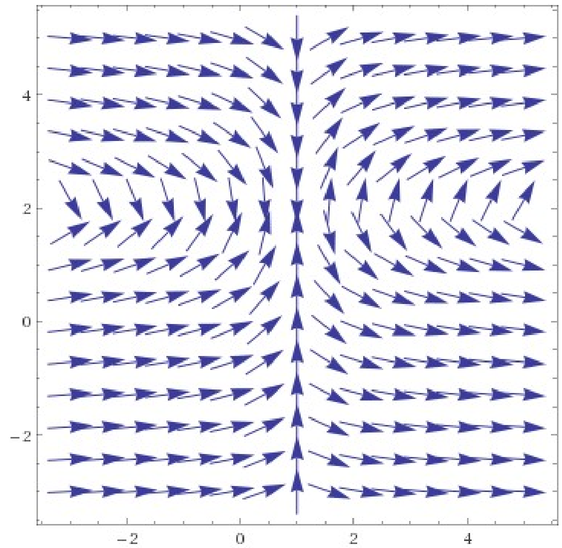
$$\frac{\partial F}{\partial y} = \frac{\partial\left(\frac{1}{(1-t)(2-y)}\right)}{\partial y} = \frac{1}{(1-t)} \frac{\partial(2-y)^{-1}}{\partial y} = \frac{1}{(1-t)(2-y)^2}$$

$\frac{\partial F}{\partial y}$ is continuous for all $t \neq 1$, $y \neq 2$

Thus the IVP $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = y_0$ has a unique solution if $t_0 \neq 1$, $y_0 \neq 2$.

Note that if $y_0 = 2$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = 2$ has two solutions if $t_0 \neq 1$ (and if we allow vertical slope in domain. Note normally our convention will be to NOT allow vertical slope in domain of solution).

Note that if $t_0 = 1$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(1) = y_0$ has no solutions.



$$(1, 1/((1-t)(2-y)))/sqrt(1+1/((1-t)(2-y))^2)$$

Solve via separation of variables: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$\int (2-y)dy = \int \frac{dt}{1-t}$

$2y - \frac{y^2}{2} = -ln|1-t| + C$

$y^2 - 4y - 2ln|1-t| + C = 0$

$y = \frac{4 \pm \sqrt{16+4(2ln|1-t|+C)}}{2} = 2 \pm \sqrt{4 + 2ln|1-t| + C}$

$$y = 2 \pm \sqrt{2ln|1-t| + C}$$

Find domain:

$2ln|1-t| + C \geq 0$ and $t \neq 1$ and $y \neq 2$

NOTE: the convention in this class to to choose largest possible connected domain where tangent line to solution is never vertical.

$2ln|1-t| \geq -C$ and $t \neq 1$ and $y \neq 2$ implies

$ln|1-t| > -\frac{C}{2}$ Note: we want to find domain for this C and thus this C can't swallow constants).

$|1-t| > e^{-\frac{C}{2}}$ since e^x is an increasing function.

$1-t < -e^{-\frac{C}{2}}$ or $1-t > e^{-\frac{C}{2}}$

$-t < -e^{-\frac{C}{2}} - 1$ or $-t > e^{-\frac{C}{2}} - 1$

Domain: $\begin{cases} t > e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 1 \\ t < -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 1. \end{cases}$

Note: Domain is much easier to determine when the ODE is linear.

Find C given $y(t_0) = y_0$**:** $y_0 = 2 \pm \sqrt{2ln|1-t_0| + C}$

$\pm(y_0 - 2) = \sqrt{2ln|1-t_0| + C}$

$(y_0-2)^2 - 2ln|1-t_0| = C$

$y = 2 \pm \sqrt{2ln|1-t| + C}$

$y = 2 \pm \sqrt{2ln|1-t| + (y_0-2)^2 - 2ln|1-t_0|}$

$y = 2 \pm \sqrt{(y_0-2)^2 + ln\frac{(1-t)^2}{(1-t_0)^2}}$

Domain: $\begin{cases} t > e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 1 \\ t < -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 1. \end{cases}$

$e^{-\frac{C}{2}} = e^{-\frac{(y_0-2)^2 - 2ln|1-t_0|}{2}} = |1-t_0|e^{-\frac{(y_0-2)^2}{2}}$

Domain: $\begin{cases} t > 1 + |1-t_0|e^{-\frac{(y_0-2)^2}{2}} & \text{if } t_0 > 1 \\ t < 1 - |1-t_0|e^{-\frac{(y_0-2)^2}{2}} & \text{if } t_0 < 1. \end{cases}$

2.4 #27b. Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when $n \neq 0, 1$ by changing it

$$y^{-n}y' + p(t)y^{1-n} = g(t)$$

when $n \neq 0, 1$ by changing it to a linear equation by substituting $v = y^{1-n}$

Example: Solve $ty' + 2t^{-2}y = 2t^{-2}y^5$

Section 2.5: Autonomous equations: $y' = f(y)$

Example: Exponential Growth/Decay

Example: population growth/radioactive decay

$$y' = ry,\ y(0) = y_0 \text{ implies } y = y_0 e^{rt}$$

$r > 0$ $\qquad\qquad$ $r < 0$

Example: Logistic growth: $y' = h(y)y$

Example: $y' = r(1 - \frac{y}{K})y$

y vs $f(y)$ $\qquad\qquad$ slope field:

Equilibrium solutions:

As $t \to \infty$, if $y > 0$, $y \to$

Solution: $y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$

Section 2.5 Autonomous equations: $y' = f(y)$

If given either differential equation $y' = f(y)$

OR direction field:

Find equilibrium solutions and determine if stable, unstable, semi-stable.

Understand what the above means.

Asymptotically stable:

Asymptotically unstable:

Asymptotically semi-stable: