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TANGLE EQUATIONS INVOLVING MONTESINOS LINKS*

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ABSTRACT

In this paper, we classify generalized Montesinos tangles and the system of unoriented tangle equations $N(U + P) = K_1$ and $N(U + R) = K_2$ is solved for a generalized Montesinos tangle U where P and R are rational tangles and at least one of K_1 and K_2 are Montesinos links.

Keywords: DNA topology, Tangle analysis, Montesinos links, Generalized Montesinos tangles

Mathematics Subject Classification 2000: 57M25, 57M27

1. Introduction

A tangle is a 3-dimensional ball with strings properly embedded in it. In 1960's, J. H. Conway introduced a tangle as a part of knot diagram while he enumerated knots and links in [3]. Tangles have been studied in knot theory and 3-manifold topology. Tangles were first used to model protein-bound DNA complexes mathematically by C. Ernst and D. W. Sumners [5]. Proteins bind to DNA segments to catalyze several biological processes that can change the topology of DNA as a result. In the tangle model, we assume the protein binding to DNA as a 3-dimensional ball and the DNA segments bound by protein as strings embedded inside the ball. As in Fig. 1, a tangle equation $N(U + P) = K_1$ is defined by a closure of the sum of two tangles U and P that equates a knot K_1. Since a protein action can be modeled by replacing one tangle P with another tangle R, this gives a system of tangle equations $N(U + P) = K_1$ and $N(U + R) = K_2$. See Fig. 1.

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The system of tangle equations $N(U+P) = K_1$ and $N(U+R) = K_2$ was solved for a generalized Montesinos tangle U where P and R are rational tangles and K_1 and K_2 are rational knots in [4]. In this paper, we solve the system of tangle equations $N(U+P) = K_1$ and $N(U+R) = K_2$ for a generalized Montesinos tangle U where P and R are rational tangles and K_1 and K_2 are either Montesinos knots/links or rational knots/links assuming at least one of them is a Montesinos knot/link.

In Sec. 1.1, we state the mathematical and biological applications of solving the system of tangle equations. In Sec. 2, basic concepts about tangles and Montesinos links are given. We classify generalized Montesinos tangles in Sec. 3. Equivalence between two systems of tangle equations is described in Sec. 4. The system of unoriented tangle equations involving Montesinos links is solved based on the classification of generalized Montesinos tangles in Sec. 5.

$N(U+P) = K_1$ and $N(U+R) = K_2$

Fig. 1. System of Tangle Equations.

1.1. *Applications*

Double branch covers of rational tangles and the sum of rational tangles are Seifert fiber space with the base surface D^2. Especially, double branch covers of rational tangles are solid tori. Double branch covers of rational knots/links and Montesinos knots/links are Seifert fiber space with the base surface S^2. Replacement of a tangle P with another tangle R in the system of tangle equations $N(U+P) = K_1$ and $N(U+R) = K_2$ corresponds to drilling out a solid torus that is the double branch cover of a rational tangle P and replacing it with another solid torus that is the double branch cover of a rational tangle R. This process is a Dehn surgery along a fiber in the Seifert fiber space as the double branch cover of K_1. Thus solving the system of tangle equations can lead us to classify the Dehn surgery in a Seifert fiber space.

In the study of protein mechanisms, it is interesting but often difficult to figure out the topology of DNA bound by protein. To explain a protein action on DNA by using the tangle equations, we think the protein-DNA complex as a tangle P and the outside of the complex as another tangle U and let the closure of the sum of P and U (Fig. 1) be equal to the topology of DNA, K_1. Since a protein action is modeled by replacing a tangle P with another tangle R, this gives a system of tangle equations $N(U+P) = K_1$ and $N(U+R) = K_2$. Thus solving this system of tangle equations can help reveal possible topological configurations of DNA bound by protein as well as the pathways of protein actions. For example, topoisomerases are involved in the crossing change of DNA knots and thus its action can be modeled by replacing $+1$ tangle with -1 tangle or vice versa as in Fig. 2. Cre recombinations can be modeled by replacing 0 tangle with ∞ tangle or vice versa as in Fig. 3. Refer Fig. 4 for basic tangles.

Fig. 2. Tangle equations modeling topoisomerase action.

Fig. 3. Tangle equations modeling *Cre* recombination.

2. Basic Concepts

2.1. *Tangles*

A 2-string tangle is a pair (B^3, t), where B^3 is a 3 dimensional ball and t is a pair of arcs properly embedded in B^3. Here, the four endpoints of the arcs are fixed at $NW = (e^{\frac{5\pi i}{4}}, 0)$, $NE = (e^{\frac{\pi i}{4}}, 0)$, $SW = (e^{-\frac{5\pi i}{4}}, 0)$ and $SE = (e^{-\frac{\pi i}{4}}, 0)$. Examples of 2-string tangles are given in Fig. 4. Two tangles are equivalent if they are ambient isotopic keeping the boundary of B^3 fixed.

A tangle is rational if it is ambient isotopic to the 0 tangle where the boundary of B^3 need not be fixed. A rational tangle can be obtained from the 0 tangle or the ∞ tangle by alternating between horizontal half twists and vertical half twists. Horizontal twists represent twists of NE and SE endpoints and vertical twists represent

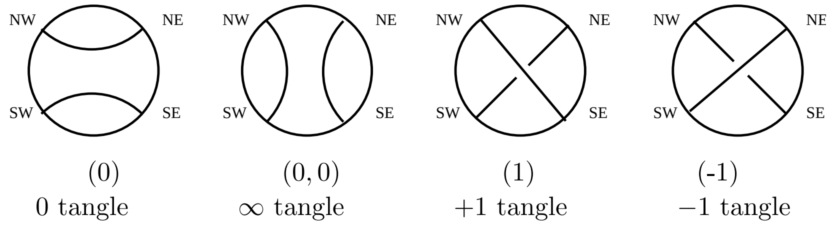


Fig. 4. 2-string tangles.

twists of SW and SE endpoints. The rational tangle obtained in this way can be expressed as a vector $(x_1, \cdots, x_n)$ where the numbers alternate between horizontal twists and vertical twists with the last number always representing horizontal twists. So if n is even, we start with vertical twists on the ∞ tangle and if n is odd, we start with horizontal twists on the 0 tangle. See Fig. 5 for examples of a rational tangle and nonrational tangles.

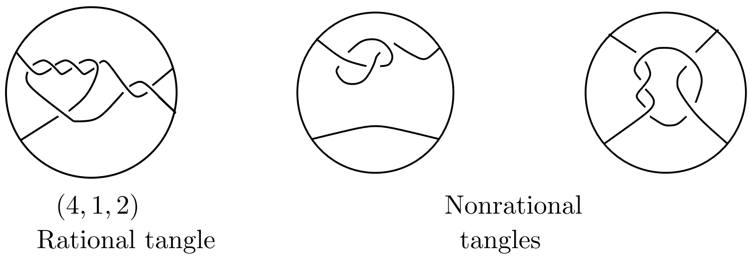


Fig. 5. Rational and NonrationalTangles.

A rational tangle $(x_1, \cdots, x_n)$ is uniquely identified by its continued fraction, $x_n + \dfrac{1}{x_{n-1} + \cdots \frac{1}{x_1}}$ [3]. A rational tangle whose corresponding continued fraction is an integer is called an integral tangle. Two rational tangles are equivalent if and only if their continued fractions are the same [3]. For example, the two tangles in Fig. 6 are equivalent. Since there are many vectors that have the same continued fractions, the vector representation for a tangle is not unique. However, every rational tangle, excluding the tangle $(0, 0)$, has a unique canonical form of vector representation $(x_1, \cdots, x_n)$, where $x_i \in Z - \{0\}$ for $1 \leq i \leq n-1$, all nonzero x_i's have the same sign and n is odd [3].

The sum of two tangles A and B, $A + B$ is obtained by connecting NE and SE endpoints of A to NW and SW endpoints of B, respectively as shown in Fig. 7.

$$2 + \frac{1}{1+\frac{1}{4}} = \frac{14}{5} = 3 + \frac{1}{-4+\frac{1}{-1}}$$

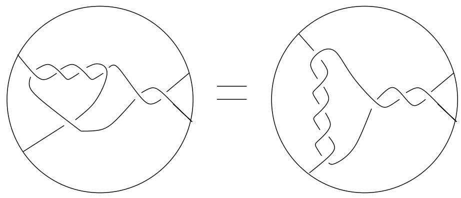


Fig. 6. Equivalent tangles.

The numerator closure of a tangle A, $N(A)$ is formed by connecting NW and NE endpoints and SW and SE endpoints as shown in Fig. 7. The numerator closure of a tangle or the sum of tangles forms a knot or link. See Fig. 7.

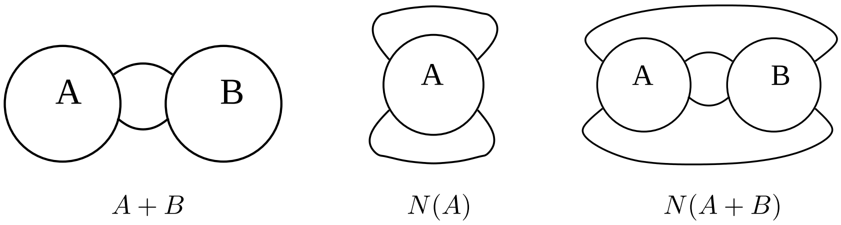


Fig. 7. Tangle sum and Numerator closures.

The circle product, $A \circ (c_1, \cdots, c_n)$ of two tangles A and $(c_1, \cdots, c_n)$ is obtained by starting with c_1 vertical (horizontal) half twists of SW and SE (NE and SE) endpoints of A and alternating between horizontal (vertical) half twists and vertical (horizontal) half twists when n is even (odd) (Fig. 8).

A generalized Montesinos tangle or generalized M-tangle is a tangle of the form $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (h_1, \cdots, h_m)$ where $\frac{a_i}{b_i}$'s are rational tangles, $\frac{a_i}{b_i} \neq \frac{1}{0}$ for $1 \leq i \leq n$ and h_j's are integers for $1 \leq j \leq m$ [4]. An *M-tangle* is a tangle of the form $\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}$ where $\frac{a_i}{b_i}$'s are rational tangles and $\frac{a_i}{b_i} \neq \frac{1}{0}$ for $1 \leq i \leq n$. A generalized M-tangle is rational if all but at most one of the $\frac{a_i}{b_i}$'s are integral. The sum of two

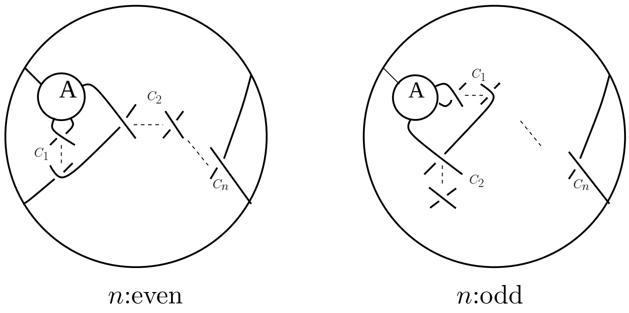


Fig. 8. Circle product $A \circ (c_1, \cdots, c_n)$.

rational tangles is rational if and only if one of the tangles is integral. In this case, $\frac{a}{b} + x = x + \frac{a}{b} = \frac{a+bx}{b}$. Also, note that $\frac{a}{b} + \frac{c}{d} = \frac{a}{b} - x + x + \frac{c}{d} = \frac{a-bx}{b} + \frac{c+dx}{d}$.

A rational knot (also called a 4-plat or 2-bridge knot/link), $N(\frac{a}{b})$, is a knot or link that can be written as the numerator closure of a rational tangle whose corresponding continued fraction is $\frac{a}{b}$. Two unoriented rational knots/links $N(\frac{a_1}{b_1})$ and $N(\frac{a_2}{b_2})$, $a_i \geq 0$ are the same if and only if $a_1 = a_2$ and $b_1 b_2^{\pm 1} \cong 1 (\text{mod } a_1)$ [2].

2.2. *Montesinos links*

A Montesinos knot/link has a projection as shown in Fig. 9 where e is an integral tangle and $\frac{a_i}{b_i}$ is a rational tangle for $i = 1, \cdots, r$ and $r \geq 3$. Here, we assume a_i and b_i are relatively prime and $0 < a_i < b_i$. This implies that $\frac{a_i}{b_i}$ is neither an integral tangle nor the infinity tangle. The above Montesinos link is written as $N(\frac{a_1}{b_1} + \cdots + \frac{a_r}{b_r} + e)$ [1,4,10].

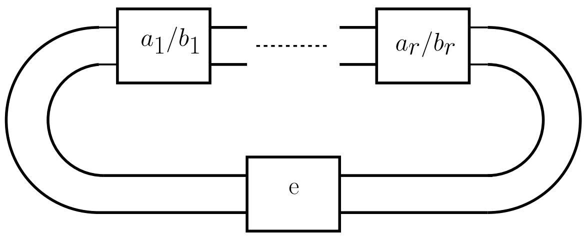


Fig. 9. Montesinos link.

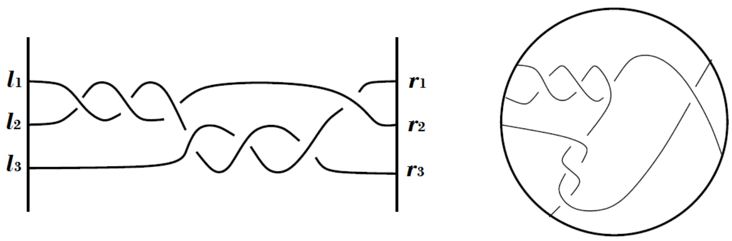
Note that Montesinos links do not include rational knots/links according to the above definition although generalized M-tangles include rational tangles.

Theorem 2.1. ***(Classification of Montesinos Links)*** *[1,2,10] Montesinos links are classified by the ordered set of fractions* $(\frac{a_1}{b_1}, \cdots, \frac{a_r}{b_r})$ *where* $r \geq 3$*, up to cyclic permutations and reversal of order, together with the integer* e *where* a_i *and* b_i *are coprime integers such that* $0 < a_i < b_i$ *for* $1 \leq i \leq r$*.*

3. Classification of generalized Montesinos Tangles

As you can see in Fig. 8, $(c_1, \cdots, c_n)$ in a circle product $A \circ (c_1, \cdots, c_n)$ can be considered as a 3-string tangle. A 3-string tangle is a 3-ball with 3 strings embedded in the ball and 6 endpoints are fixed on the boundary of the ball (Fig. 10 (b)). The 3-string tangles used in the circle product will be regarded as 3-braids which will be used to classify generalized Montesinos tangles in theorem 3.9.

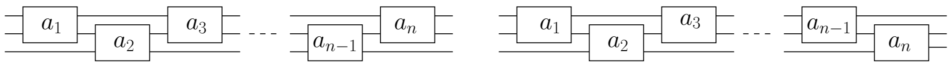
A 3-braid is a set of 3 strings which are attached to vertical bars at the left and at the right as in Fig. 10 (a). Each string always heads to the right as we move along the string from the left vertical bar to the right vertical bar. A 3-braid can be considered as a special case of a 3-string tangle as in Fig. 10.



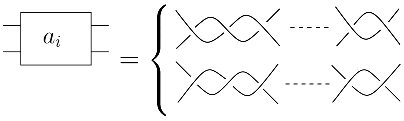
(a) $B(3,3,1)$ (b)

Fig. 10. Converting a 3-braid into a 3-string tangle; (a) a 3-braid and (b) the corresponding 3-string tangle.

We denote a 3-braid by $B(a_1, \cdots, a_n)$ as in Fig. 11 where $a_i \in Z$ [6,7]. Since $(g_1, \cdots, g_k)$ in $A \circ (g_1, \cdots, g_k)$ can be considered as a 3-braid by placing vertical twists horizontally and moving the SW endpoints of A and $A \circ (g_1, \cdots, g_k)$ to the east sides of A and $A \circ (g_1, \cdots, g_k)$ respectively as in Fig. 12, a circle product can be related to the sum between a 2-string tangle and a 3-braid as follows: first, define the sum between a 2-string tangle A and a 3-braid B, $A +_{tb} B$ by connecting NE, SE and SW endpoints of A to l_1, l_2 and l_3 of B, respectively. Then we can represent $A \circ (g_1, \cdots, g_k)$ as $A +_{tb} B(g_1, \cdots, g_k)$ where A is a 2-string tangle and $B(g_1, \cdots, g_k)$ is the 3-braid corresponding to $(g_1, \cdots, g_k)$ in $A \circ (g_1, \cdots, g_k)$.



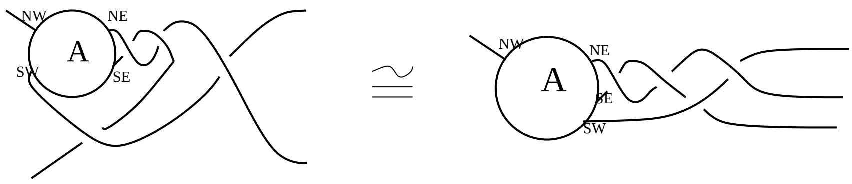
(a) n odd (b) n even

where 

if a_i is positive(negative) and i is odd(even)

if a_i is negative(positive) and i is odd(even)

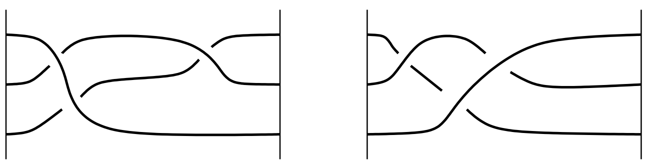
Fig. 11. $B(a_1, \cdots, a_n)$.



(a)$A \circ (2, 1, 1)$ (b)$A +_{tb} B(2, 1, 1)$

Fig. 12. Circle product as the sum between a 2 string tangle and a 3-braid; (a) a circle product and (b) the corresponding sum.

The braid sum, $A +_b B$ of two 3-braids A and B is defined by connecting r_1, r_2, r_3 of A to l_1, l_2, l_3 of B, respectively [6,7]. Every 3-braid $B(h_1, \cdots, h_m)$ can be represented as a standard diagram $B(g_1, \cdots, g_k) +_b sE$ for $s \in Z$ where g_i's have the same sign and E is shown in Fig. 13 (a) [6]. Note that $sE = E +_b \cdots +_b E$ (s-times) if $s > 0$ and $sE = (-E) +_b \cdots +_b (-E)$ (s-times) if $s < 0$.



(a)$E = B(1, -1, 1)$ (b)$-E = B(-1, 1, -1)$

Fig. 13. Braids; (a) E and (b) $-E$.

Theorem 3.1.
Let $B(h_1, \cdots, h_m) = B(g_1, \cdots, g_k) +_b sE$ for $s \in Z$ where g_i's have the same sign and E is shown in Fig. 13 (a). Then a generalized M-tangle can be represented as

follows:

$$(\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n})\circ(h_1,\cdots,h_m)=\begin{cases}(\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n})\circ(g_1,\cdots,g_k) & \text{if } s \text{ is even}\\ (\frac{-b_1}{a_1}*\cdots*\frac{-b_n}{a_n})\circ(-g_1,\cdots,-g_k,0) & \text{if } s \text{ is odd}\end{cases}$$

where $A*B$ is the vertical sum of tangles A and B defined by connecting SW and SE endpoints of A with NW and NE endpoints of B, respectively [8].

Proof. $(\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n})\circ(h_1,\cdots,h_m)=(\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n})+_{tb}B(h_1,\cdots,h_m)$ $=(\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n})+_{tb}[B(g_1,\cdots,g_k)+_b sE]=[(\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n})+_{tb}B(g_1,\cdots,g_k)]+_{tb}sE$ $=[(\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n})\circ(g_1,\cdots,g_k)]\circ(1,-1,1)\circ(1,-1,1)\circ\cdots\circ(1,-1,1)$.
Let $A=(\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n})\circ(g_1,\cdots,g_k)$. For $T=A\circ(1,-1,1)\circ(1,-1,1)\circ\cdots\circ(1,-1,1)$, let $r_A(T)$ be the rotation of a tangle T by 180° about the line from the NW and the SE of A and $r(T)$ be the rotation of a tangle T by 180° about the line from the NW and the SE of T. Note that $r_A(A\circ(1,-1,1))=r(A)$ and $r_A^2(A\circ(1,-1,1)\circ(1,-1,1))=A$. Thus by induction, $A\circ(1,-1,1)\circ(1,-1,1)\circ\cdots\circ(1,-1,1)$ is deformed into A if s is even and $r(A)$ if s is odd. Note that $r(A)=r((\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n})\circ(g_1,\cdots,g_k))=(r(\frac{a_1}{b_1})*\cdots*r(\frac{a_n}{b_n}))\circ(-g_1,\cdots,-g_k,0)=(\frac{-b_1}{a_1}*\cdots*\frac{-b_n}{a_n})\circ(-g_1,\cdots,-g_k,0)$. □

The following lemmas and the Euler bracket function are used to prove proposition 3.6, 3.7 and 3.8.

Lemma 3.2. *$N(A+C)=N(C+A)$ where A and C are arbitrary tangles.*

Lemma 3.3. *$N(A\circ(c_1,\cdots,c_n)+B)=N(A+B\circ(c_n,\cdots,c_1))$ if n is odd and B is invariant under 180° rotations about both the x and y axes.*

Proof. If $n=1$, $N(A\circ(c_1)+B)=N(A+B\circ(c_1))$ since B is invariant under 180° rotation about the x axis. Assume that $N(A\circ(c_1,\cdots,c_n)+B)=N(A+B\circ(c_n,\cdots,c_1))$ when $n=2k-1$ for $k\geq 1$. Then $N(A\circ(c_1,\cdots,c_{2k+1})+B)=N(A\circ(c_1,\cdots,c_{2k},0)+B\circ(c_{2k+1}))=N(A\circ(c_1,\cdots,c_{2k-1})\circ(c_{2k},0)+B\circ(c_{2k+1}))=N(A\circ(c_1,\cdots,c_{2k-1})+B\circ(c_{2k+1})\circ(c_{2k},0))=N(A\circ(c_1,\cdots,c_{2k-1})+B\circ(c_{2k+1},c_{2k},0))$. Since B is invariant under 180° rotations about both the x and y axes, so are $B\circ(c_{2k+1})$ and then $B\circ(c_{2k+1})\circ(c_{2k},0)$. Thus by the induction hypothesis, $N(A\circ(c_1,\cdots,c_{2k-1})+B\circ(c_{2k+1},c_{2k},0))=N(A+B\circ(c_{2k+1},c_{2k},0)\circ(c_{2k-1},\cdots,c_1))=N(A+B\circ(c_{2k+1},c_{2k},c_{2k-1},\cdots,c_1))$. By induction, we have the result. □

Note that since a rational tangle is invariant under 180° rotations about both the x and y axes, lemma 3.3 holds for a rational tangle B.

Lemma 3.4. *[4] $(d_1,\cdots,d_m)\circ(c_1,\cdots,c_n)=(d_1,\cdots,d_m+c_1,\cdots,c_n)$ if n is odd.*

The Euler bracket function, $E[x_1, \cdots, x_n]$ equals the sum of the products obtained from the product $1 \cdot x_1 \cdot \cdots \cdot x_n$ by omitting zero or more disjoint pairs of consecutive $x_i x_{i+1}$ from the product [9]. The Euler bracket functions satisfy the followings.

Lemma 3.5. *[9]*

(1) *If* $n = 0$, *then* $E[x_1, \cdots, x_n] = E[] = 1$.
(2) *If* $n < 0$, *then* $E[x_1, \cdots, x_n] = 0$.
(3) $E[x_1, \cdots, x_n] = E[x_n, \cdots, x_1]$.
(4) *For* $n \geq 1$,

(a) $E[x_1, \cdots, x_n] = x_1 E[x_2, \cdots, x_n] + E[x_3, \cdots, x_n]$ $= x_n E[x_1, \cdots, x_{n-1}] + E[x_1, \cdots, x_{n-2}]$.

(b) $[x_n, \cdots, x_1] = \dfrac{E[x_1, \cdots, x_n]}{E[x_1, \cdots, x_{n-1}]} = \dfrac{E[x_n, \cdots, x_1]}{E[x_{n-1}, \cdots, x_1]}$ *where* $[x_n, \cdots, x_1] = x_n + \dfrac{1}{x_{n-1} + \cdots + \frac{1}{x_1}}$.

(c) *Let* $a = E[x_1, \cdots, x_n]$,$b = E[x_1, \cdots, x_{n-1}]$. *If* $y = (-1)^{n+1} E[x_2, \cdots, x_{n-1}]$ *and* $x = (-1)^{n+1} E[x_2, \cdots, x_n]$, *then* $bx - ay = 1$.

(5) $[c_1, \cdots, c_n + d_m, \cdots, d_1] = \frac{E[c_1,\cdots,c_n]E[d_1,\cdots,d_{m-1}]+E[c_1,\cdots,c_{n-1}]E[d_1,\cdots,d_m]}{E[c_2,\cdots,c_n]E[d_1,\cdots,d_{m-1}]+E[c_2,\cdots,c_{n-1}]E[d_1,\cdots,d_m]}$.

Proposition 3.6. *Suppose that* $0 < a_i < b_i$ *for* $1 \leq i \leq n$, h_j*'s have the same sign for all* j, $h_j \neq 0$ *for* $2 \leq j \leq t-1$ *and* t *is odd. For* $n \geq 2$, $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (h_1, \cdots, h_t) = (\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m}) \circ (k_1, \cdots, k_s)$ *where* $0 < c_i < d_i$ *for* $1 \leq i \leq m$, k_j*'s have the same sign for all* j, $k_j \neq 0$ *for* $2 \leq j \leq s-1$ *and* s *is odd iff (a)* $n = m$ *and* $\frac{a_i}{b_i} = \frac{c_i}{d_i}$ *for all* i *and (b)* $t = s$ *and* $h_j = k_j$ *for all* j.

Proof. ($\Rightarrow$) Suppose that $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (h_1, \cdots, h_t) = (\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m}) \circ (k_1, \cdots, k_s)$. Then $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) = (\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m}) \circ (k_1, \cdots, k_s) \circ (-h_t, \cdots, -h_1) = (\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m}) \circ (k_1, \cdots, k_s - h_t, \cdots, -h_1)$ by lemma 3.4. We can choose a rational tangle $\frac{x}{y}$ such that $0 < x < y$, x and y are coprime and $\frac{x}{y} \neq \frac{c_i}{d_i}$ for any i. Then $N(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n} + \frac{x}{y}) = N((\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m}) \circ (k_1, \cdots, k_s - h_t, \cdots, -h_1) + \frac{x}{y}) = N(\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m} + \frac{x}{y} \circ (-h_1, \cdots, -h_t + k_s, \cdots, k_1))$ by lemma 3.3. Since $\frac{x}{y} \neq \frac{c_i}{d_i}$ and $\frac{x}{y}$ is not integral, by theorem 2.1, $\frac{x}{y} = \frac{x}{y} \circ (-h_1, \cdots, -h_t + k_s, \cdots, k_1)$. If we represent the rational tangle $\frac{x}{y}$ as the canonical form $(l_1, \cdots, l_u)$, then $\frac{x}{y} \circ (-h_1, \cdots, -h_t + k_s, \cdots, k_1) = (l_1, \cdots, l_u) \circ$

$(-h_1, \cdots, -h_t+k_s, \cdots, k_1) = (l_1, \cdots, l_u-h_1, \cdots, -h_t+k_s, \cdots, k_1) = \frac{x}{y}$. Since two tangles $\frac{x}{y} \circ (-h_1, \cdots, -h_t + k_s, \cdots, k_1)$ and $\frac{x}{y}$ are the same, their continued fractions are the same. That is, by lemma 3.5 (5), $[k_1, \cdots, k_s-h_t, \cdots, -h_1+l_u, \cdots, l_1] = \frac{E[k_1, \cdots, k_s - h_t, \cdots, -h_1]E[l_1, \cdots, l_{u-1}] + E[k_1, \cdots, k_s - h_t, \cdots, -h_2]E[l_1, \cdots, l_u]}{E[k_2, \cdots, k_s - h_t, \cdots, -h_1]E[l_1, \cdots, l_{u-1}] + E[k_2, \cdots, k_s - h_t, \cdots, -h_2]E[l_1, \cdots, l_u]}$ $= \frac{E[k_1, \cdots, k_s - h_t, \cdots, -h_1]y + E[k_1, \cdots, k_s - h_t, \cdots, -h_2]x}{E[k_2, \cdots, k_s - h_t, \cdots, -h_1]y + E[k_2, \cdots, k_s - h_t, \cdots, -h_2]x} = \frac{x}{y}$ since $\frac{x}{y} = [l_u, \cdots, l_1] = \frac{E[l_1, \cdots, l_u]}{E[l_1, \cdots, l_{u-1}]}$ and x and y are coprime.

Let $E[k_1, \cdots, k_s-h_t, \cdots, -h_1] = a, E[k_1, \cdots, k_s-h_t, \cdots, -h_2] = b, E[k_2, \cdots, k_s-h_t, \cdots, -h_1] = a'$ and $E[k_2, \cdots, k_s - h_t, \cdots, -h_2] = b'$. Then $\frac{ay + bx}{a'y + b'x} = \frac{x}{y}$ where $(a, b) = 1$ and $(x, y) = 1$. This implies that $ay + bx = kx$ for some integer k. Then $ay = (k - b)x$. Since $(x, y) = 1$, $x|a$ for any x such that $0 < x < y$, $(x, y) = 1$ and $\frac{x}{y} \neq \frac{c_i}{d_i}$ for any i. Thus $a = E[k_1, \cdots, k_s - h_t, \cdots, -h_1] = 0$ and so $\frac{E[k_1, \cdots, k_s - h_t, \cdots, -h_1]}{E[k_2, \cdots, k_s - h_t, \cdots, -h_1]} = [k_1, \cdots, k_s - h_t, \cdots, -h_1] = 0$.

Since $[k_1, \cdots, k_s - h_t, \cdots, -h_1] = \frac{E[k_1, \cdots, k_s]E[-h_1, \cdots, -h_{t-1}] + E[k_1, \cdots, k_{s-1}]E[-h_1, \cdots, -h_t]}{E[k_2, \cdots, k_s]E[-h_1, \cdots, -h_{t-1}] + E[k_2, \cdots, k_{s-1}]E[-h_1, \cdots, -h_t]}$ $= \frac{E[k_1, \cdots, k_s]E[h_1, \cdots, h_{t-1}] - E[k_1, \cdots, k_{s-1}]E[h_1, \cdots, h_t]}{E[k_2, \cdots, k_s]E[h_1, \cdots, h_{t-1}] - E[k_2, \cdots, k_{s-1}]E[h_1, \cdots, h_t]} = \frac{0}{1}$ by lemma 3.5, $E[k_1, \cdots, k_s]E[h_1, \cdots, h_{t-1}] - E[k_1, \cdots, k_{s-1}]E[h_1, \cdots, h_t] = 0$. Therefore $\frac{E[k_1, \cdots, k_s]}{E[k_1, \cdots, k_{s-1}]} = \frac{E[h_1, \cdots, h_t]}{E[h_1, \cdots, h_{t-1}]}$ which implies $(k_1, \cdots, k_s) = (h_1, \cdots, h_t)$ as 2-string rational tangles.

(1) Assume that $h_1 \neq 0$. If $k_1 \neq 0$, then $(k_1, \cdots, k_s)$ and $(h_1, \cdots, h_t)$ are both unique canonical forms of a rational tangle since t and s are odd, all h_j's have the same sign, all k_j's have the same sign, $h_j \in Z-\{0\}$ for $1 \leq j \leq t-1$ and $k_j \in Z-\{0\}$ for $1 \leq j \leq s-1$. Since $(k_1, \cdots, k_s) = (h_1, \cdots, h_t)$ and they are unique, $t = s$ and $h_j = k_j$ for all j. If $k_1 = 0$, then $(h_1, \cdots, h_t) = (0, k_2, k_3, \cdots, k_s) = (k_3, \cdots, k_s)$. Then $t = s - 2$ and $h_i = k_{i+2}$ for $i = 1, 2, \cdots, t$ since h_i and k_j are nonzero for $i = 1, \cdots, t$ and $j = 3, \cdots, s$, h_i's have the same sign and k_j's have the same sign. Thus $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (h_1, \cdots, h_t) = (\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m}) \circ (0, k_2, h_1, \cdots, h_t)$ which implies that $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) = (\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m}) \circ (0, k_2) = (\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m}) \circ (k_2)$. Then $N(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n} + \frac{x}{y}) = N((\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m}) \circ (k_2) + \frac{x}{y}) = N(\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m} + \frac{x}{y} \circ (k_2))$ by lemma 3.3 for a nonintegral rational tangle $\frac{x}{y}$ such that $0 < x < y$, x and y are coprime and $\frac{x}{y} \neq \frac{c_i}{d_i}$ for any i. By theorem 2.1, $\frac{x}{y} = \frac{x}{y} \circ (k_2) = \frac{x}{y} + k_2 = \frac{x + k_2 y}{y}$. Thus $x = x + k_2 y$ and so $k_2 = 0$. This contradicts the hypothesis.

(2) Assume that $h_1 = 0$. If $k_1 \neq 0$, then by the similar argument as in (1), $h_2 = 0$ which contradicts the hypothesis. If $k_1 = 0$, then $(h_3, \cdots, h_t) = (k_3, \cdots, k_s)$ as 2-string rational tangles and both $(h_3, \cdots, h_t)$ and $(k_3, \cdots, k_s)$ are canonical representation of rational tangles. Thus $t = s$ and $h_j = k_j$ for $3 \leq j \leq t$. Then $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (0, h_2, h_3, \cdots, h_t) = (\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m}) \circ (0, k_2, h_3, \cdots, h_t)$ which implies that $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) = (\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m}) \circ (0, k_2 - h_2) = (\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m}) \circ (k_2 - h_2)$. By the same argument as in (1), $k_2 = h_2$.

Since $s = t$ and $h_j = k_j$ for all $1 \leq j \leq t$, $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (h_1, \cdots, h_t) = (\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m}) \circ (k_1, \cdots, k_s)$ implies that $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) = (\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m})$. Now, choose another nonintegral rational tangle $\frac{z}{w} (\neq \frac{x}{y})$ such that $0 < z < w$, $(z, w) = 1$ and $\frac{z}{w} \neq \frac{c_i}{d_i}$ for any i. Then $N(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n} + \frac{x}{y} + \frac{z}{w}) = N(\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m} + \frac{x}{y} + \frac{z}{w})$. Then by theorem 2.1, $\frac{a_i}{b_i} = \frac{c_i}{d_i}$ for all i.

($\Leftarrow$) It is trivial. □

Proposition 3.7. *Suppose that $0 < a_i < b_i$ for $1 \leq i \leq n$, h_j's have the same sign for all j, $h_j \neq 0$ for $2 \leq j \leq t-1$ and t is odd. For $n \geq 2$, $(\frac{-b_1}{a_1} * \cdots * \frac{-b_n}{a_n}) \circ (-h_1, \cdots, -h_t, 0) = (\frac{-d_1}{c_1} * \cdots * \frac{-d_m}{c_m}) \circ (-k_1, \cdots, -k_s, 0)$ where $0 < c_i < d_i$, k_j's have the same sign for all j, $k_j \neq 0$ for $2 \leq j \leq s-1$ and s is odd iff (a) $n = m$ and $\frac{a_i}{b_i} = \frac{c_i}{d_i}$ for all i and (b) $t = s$ and $h_j = k_j$ for all j.*

Proof. ($\Rightarrow$) Suppose $(\frac{-b_1}{a_1} * \cdots * \frac{-b_n}{a_n}) \circ (-h_1, \cdots, -h_t, 0) = (\frac{-d_1}{c_1} * \cdots * \frac{-d_m}{c_m}) \circ (-k_1, \cdots, -k_s, 0)$. By rotating both these tangles about the lines connecting NW and SE endpoints of $(\frac{-b_1}{a_1} * \cdots * \frac{-b_n}{a_n}) \circ (-h_1, \cdots, -h_t, 0)$ and $(\frac{-d_1}{c_1} * \cdots * \frac{-d_m}{c_m}) \circ (-k_1, \cdots, -k_s, 0)$ respectively, we have $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (h_1, \cdots, h_t) = (\frac{c_1}{d_1} + \cdots + \frac{c_m}{d_m}) \circ (k_1, \cdots, k_s)$. By proposition 3.6, we have the result.

($\Leftarrow$) It is clear. □

Proposition 3.8. *Suppose that $0 < a_i < b_i$ for $1 \leq i \leq n$, h_j's have the same sign for all j, $h_j \neq 0$ for $2 \leq j \leq t-1$ and t is odd. If $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (h_1, \cdots, h_t) = (\frac{-d_1}{c_1} * \cdots * \frac{-d_m}{c_m}) \circ (-k_1, \cdots, -k_s, 0)$ where $0 < c_i < d_i$, k_j's have the same sign for all j, $k_j \neq 0$ for $2 \leq j \leq s-1$ and s is odd, then $n = m = 1$. That is, $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (h_1, \cdots, h_t)$ is a rational tangle.*

Proof. Suppose $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (h_1, \cdots, h_t) = (\frac{-d_1}{c_1} * \cdots * \frac{-d_m}{c_m}) \circ (-k_1, \cdots, -k_s, 0)$. Then $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (h_1, \cdots, h_t, k_s, \cdots, k_1, 0) = (\frac{-d_1}{c_1} * \cdots * \frac{-d_m}{c_m})$. Taking numerator closure of both sides, $N((\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (h_1, \cdots, h_t, k_s, \cdots, k_1, 0)) = N(\frac{-d_1}{c_1} * \cdots * \frac{-d_m}{c_m}) = D(\frac{c_1}{d_1})\sharp \cdots \sharp D(\frac{c_m}{d_m})$. Since $N((\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (h_1, \cdots, h_t, k_s, \cdots, k_1, 0)) = N((\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) + (0, k_1, \cdots, k_s, h_t, \cdots, h_1))$ is either a rational link or a Montesinos link, $m = 1$. Then $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (h_1, \cdots, h_t) = \frac{-d_1}{c_1} \circ (-k_1, \cdots, -k_s, 0)$ which is a rational tangle. Thus $n = 1$. That is, $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (h_1, \cdots, h_t) = \frac{a_1}{b_1} \circ (h_1, \cdots, h_t)$. □

Propositions 3.6, 3.7 and 3.8 give the following classification of generalized Montesinos tangles which are not rational tangles.

Theorem 3.9. ***(Classification of Generalized Montesinos Tangles)*** *A generalized Montesinos tangle which is not rational is uniquely represented as one of the following:*
(1) $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (h_1, \cdots, h_t)$ *when* $n \geq 2$*,*
(2) $(\frac{-b_1}{a_1} * \cdots * \frac{-b_n}{a_n}) \circ (-h_1, \cdots, -h_t, 0)$ *when* $n \geq 2$*.*
where $0 < a_i < b_i$*,* t *is odd,* h_j*'s have the same sign for all* j *and* $h_j \neq 0$ *for* $2 \leq j \leq t-1$

Proof. By theorems 3.1, a generalized Montesinos tangle has the form $(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}) \circ (h_1, \cdots, h_t)$ or $(\frac{-b_1}{a_1} * \cdots * \frac{-b_n}{a_n}) \circ (-h_1, \cdots, -h_t, 0)$ where $0 < a_i < b_i$, t is odd, h_j's have the same sign for all j and $h_j \neq 0$ for $2 \leq j \leq t-1$. By proposition 3.6 and 3.7, this is unique. □

The sum of two rational tangles need not be rational but the numerator closure of the sum of two rational tangles is a rational knot.

Lemma 3.10. *[5]* $N(\frac{j}{p} + \frac{t}{w}) = N(\frac{jw + pt}{dw + qt})$ *where* d *and* q *are any integers such that* $pd - qj = 1$*.*

Lemma 3.11. *[4] If* $N(\frac{j}{p} + \frac{f}{g}) = N(\frac{a}{b})$*, then* $\frac{f}{g} = \frac{da - jb'}{pb' - qa}$ *for some integers* d*,* q*, and* b' *such that* $pd - qj = 1$*,* $b'b^{\pm 1} = 1$ *mod* a*.*

4. Equivalent moves

In this section, we explain the equivalence between two systems of tangle equations.

Definition 4.1. [4] If there is a solution for U such that $N(U+B) = K_1$ and $N(U+E) = K_2$, then K_2 is said to be obtained from K_1 by a (B, E)-move.

Definition 4.2. [4] A (B, E)-move is said to be equivalent to a (B', E')-move if there exists a solution for U such that $N(U+B) = K_1$ and $N(U+E) = K_2$ if and only if there exists a solution for U' such that $N(U'+B') = K_1$ and $N(U'+E') = K_2$. The above two systems of tangle equations are said to be equivalent.

For example, $(+1, -1)$-move is equivalent to $(0, -2)$-move as shown in Fig. 14 and the corresponding equivalent systems of tangle equations are given in Fig. 15.

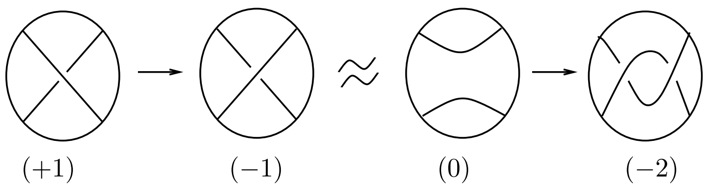


Fig. 14. Equivalent moves.

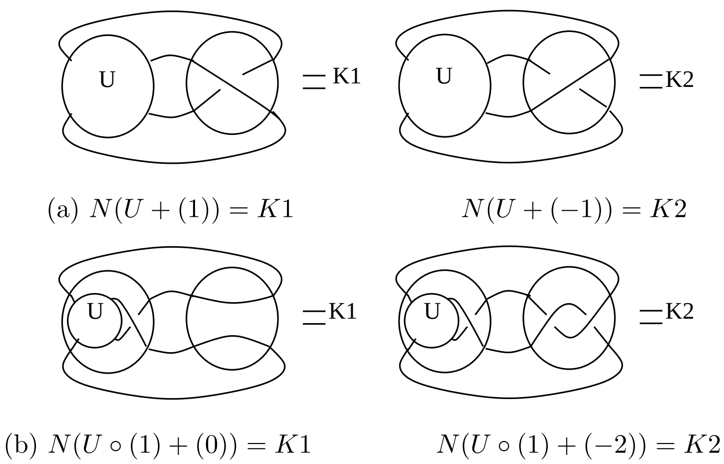


Fig. 15. Equivalent systems of tangle equations.

In the next theorem, we give relations between (P, R)-move and $(0, \frac{x}{y})$-move for rational tangles P and R and thus equivalent systems of tangle equations.

Theorem 4.3. *[4] Suppose that P and R are rational tangles where $P = (c_1, \cdots, c_n)$ for n odd. If $\frac{x}{y} = R \circ (-c_n, \cdots, -c_1)$ and $U' = U \circ (c_n, \cdots, c_1)$ (or equivalently, if $R = \frac{x}{y} \circ (c_1, \cdots, c_n)$ and $U = U' \circ (-c_1, \cdots, -c_n)$), then for any knots K_1 and K_2, the system of tangle equations $N(U + P) = K_1$ and $N(U + R) = K_2$ and the system of tangle equations $N(U' + \frac{0}{1}) = K_1$ and $N(U' + \frac{x}{y}) = K_2$ are equivalent.*

Proof. $N(U + P) = N(U + (c_1, \cdots, c_n)) = N(U \circ (c_n, \cdots, c_1) + \frac{0}{1})$ by lemma 3.3 which is equal to $N(U' + \frac{0}{1})$ where $U' = U \circ (c_n, \cdots, c_1)$. Also, $N(U + R) = N(U \circ (c_n, \cdots, c_1) \circ (-c_1, \cdots, -c_n) + R) = N(U \circ (c_n, \cdots, c_1) + R \circ (-c_n, \cdots, -c_1))$ by lemma 3.3 which is equal to $N(U' + \frac{x}{y})$ where $U' = U \circ (c_n, \cdots, c_1)$ and $\frac{x}{y} = R \circ (-c_n, \cdots, -c_1)$. Note that $U' = U \circ (c_n, \cdots, c_1)$ if and only if $U = U' \circ (-c_1, \cdots, -c_n)$ and $\frac{x}{y} = R \circ (-c_n, \cdots, -c_1)$ if and only if $R = \frac{x}{y} \circ (c_1, \cdots, c_n)$. □

Corollary 4.4. *For any knots K_1 and K_2, there exists a generalized Montesinos tangle U such that $N(U + P) = K_1$ and $N(U + R) = K_2$ where P and R are rational knots if and only if there exists a generalized Montesinos tangle U' such that $N(U' + \frac{0}{1}) = K_1$ and $N(U' + \frac{x}{y}) = K_2$.*

Proof. By theorem 4.3, if $P = (c_1, \cdots, c_n)$ for n odd, then $U' = U \circ (c_n, \cdots, c_1)$ and $\frac{x}{y} = R \circ (-c_n, \cdots, -c_1)$ (or equivalently, $U = U' \circ (-c_1, \cdots, -c_n)$ and $R = \frac{x}{y} \circ (c_1, \cdots, c_n)$). Moreover, $U = U' \circ (-c_1, \cdots, -c_n)$ is a generalized Montesinos tangle if and only if $U' = U \circ (c_n, \cdots, c_1)$ is a generalized Montesinos tangle. □

Thus solving the system of tangle equations $N(U' + \frac{0}{1}) = K_1$ and $N(U' + \frac{x}{y}) = K_2$ gives us the solutions to the system of tangle equations $N(U + P) = K_1$ and $N(U + R) = K_2$ and vice versa.

5. Solving tangle equations

First, the system of tangle equations is solved when $s \geq 3$ and $t \geq 3$. That is, the righthand side of equations (1) and (2) are both Montesinos links.

Theorem 5.1. *Suppose that a_i, b_i, e_1, x, y are integers and $0 < a_i < b_i$ for $1 \leq i \leq s$. For $s, t \geq 3$,*

$$N(U + \frac{0}{1}) = N(\frac{a_1}{b_1} + \cdots + \frac{a_s}{b_s} + e_1) \quad (1)$$

$$\text{and } N(U + \frac{x}{y}) = N(\frac{z_1}{v_1} + \cdots + \frac{z_t}{v_t} + e_2) \quad (2)$$

where z_j, v_j, e_2 are integers and $0 < z_j < v_j$ for $1 \leq j \leq t$

and $U = (\frac{c_1}{d_1} + \cdots + \frac{c_n}{d_n}) \circ (h_1, \cdots, h_m)$ is a generalized M-tangle.

if and only if for $s, t \geq 3$ and for some $1 \leq j_1 \leq s$,

(1) *If $m = 1$, then $U = (\frac{a_{j_1}}{b_{j_1}} + \frac{a_{j_1 \pm 1}}{b_{j_1 \pm 1}} + \cdots + \frac{a_{j_1 \mp 2}}{b_{j_1 \mp 2}} + \frac{a_{j_1 \mp 1}}{b_{j_1 \mp 1}}) \circ (e_1)$ and $N(\frac{z_1}{v_1} + \cdots + \frac{z_t}{v_t} + e_2) = N(\frac{a_{j_1}}{b_{j_1}} + \frac{a_{j_1 \pm 1}}{b_{j_1 \pm 1}} + \cdots + \frac{a_{j_1 \mp 2}}{b_{j_1 \mp 2}} + \frac{a_{j_1 \mp 1}}{b_{j_1 \mp 1}} + \frac{x}{y} + e_1)$ where $\frac{x}{y} = e_2 - e_1$ and $t = s$ if $\frac{x}{y}$ is an integer and $\frac{x}{y} = \frac{z_k}{v_k} + e_2 - e_1$ for some k and $t = s + 1$ if $\frac{x}{y}$ is not an integer.*

(2) *If $m = 3$ and $h_m = 0$, then $U = (\frac{a_{j_1}}{b_{j_1}} + \frac{a_{j_1 \pm 1}}{b_{j_1 \pm 1}} + \cdots + \frac{a_{j_1 \mp 2}}{b_{j_1 \mp 2}} + \frac{a_{j_1 \mp 1}}{b_{j_1 \mp 1}}) \circ (e_1, h_2, 0)$ and $N(\frac{z_1}{v_1} + \cdots + \frac{z_t}{v_t} + e_2) = N(\frac{a_{j_1}}{b_{j_1}} + \frac{a_{j_1 \pm 1}}{b_{j_1 \pm 1}} + \cdots + \frac{a_{j_1 \mp 2}}{b_{j_1 \mp 2}} + \frac{a_{j_1 \mp 1}}{b_{j_1 \mp 1}} + \frac{x}{h_2 x + y} + e_1)$ where $h_2 = \frac{1}{e_2 - e_1} - \frac{y}{x}$ and $t = s$ if $\frac{x}{h_2 x + y}$ is an integer and $h_2 = \frac{v_k}{z_k + (e_2 - e_1) v_k} - \frac{y}{x}$ for some k and $t = s + 1$ if $\frac{x}{h_2 x + y}$ is not an integer.*

(3) *If $m > 3$ or if $m = 3$ and $h_m \neq 0$, then assuming that h_i's have the same sign and $h_i \neq 0$ for $2 \leq i \leq m - 1$,*

(a) *$U = (\frac{a_{j_1}}{b_{j_1}} + \frac{a_{j_1 \pm 1}}{b_{j_1 \pm 1}} + \cdots + \frac{a_{j_1 \mp 3}}{b_{j_1 \mp 3}} + \frac{a_{j_1 \mp 2}}{b_{j_1 \mp 2}}) \circ (h_1, \cdots, h_m)$ and $N(\frac{z_1}{v_1} + \cdots + \frac{z_t}{v_t} + e_2) = N(\frac{a_{j_1}}{b_{j_1}} + \frac{a_{j_1 \pm 1}}{b_{j_1 \pm 1}} + \cdots + \frac{a_{j_1 \mp 3}}{b_{j_1 \mp 3}} + \frac{a_{j_1 \mp 2}}{b_{j_1 \mp 2}} + \frac{xE[h_1, \cdots, h_{m-1}] + yE[h_1, \cdots, h_m]}{xE[h_2, \cdots, h_{m-1}] + yE[h_2, \cdots, h_m]})$ where $\frac{E[h_1, \cdots, h_m]}{E[h_2, \cdots, h_m]} = \frac{a_{j_1 \mp 1}}{b_{j_1 \mp 1}} + e_1$ for some j_1.*

(b) *$U = (\frac{-b_{j_1}}{a_{j_1}} * \frac{-b_{j_1 \pm 1}}{a_{j_1 \pm 1}} * \cdots * \frac{-b_{j_1 \mp 3}}{a_{j_1 \mp 3}} * \frac{-b_{j_1 \mp 2}}{a_{j_1 \mp 2}}) \circ (-h_1, \cdots, -h_m, 0)$ and $N(\frac{z_1}{v_1} + \cdots + \frac{z_t}{v_t} + e_2) = N(\frac{a_{j_1}}{b_{j_1}} + \frac{a_{j_1 \pm 1}}{b_{j_1 \pm 1}} + \cdots + \frac{a_{j_1 \mp 3}}{b_{j_1 \mp 3}} + \frac{a_{j_1 \mp 2}}{b_{j_1 \mp 2}} + \frac{yE[h_1, \cdots, h_{m-1}] - xE[h_1, \cdots, h_m]}{yE[h_2, \cdots, h_{m-1}] - xE[h_2, \cdots, h_m]})$ where $h_2 \neq \pm 1$ and $\frac{E[h_1, \cdots, h_{m-1}]}{E[h_2, \cdots, h_{m-1}]} = \frac{a_{j_1 \mp 1}}{b_{j_1 \mp 1}} + e_1$ for some j_1.*

(c) *$m = 3$ and $U = (\frac{-b_{j_1}}{a_{j_1}} * \frac{-b_{j_1 \pm 1}}{a_{j_1 \pm 1}} * \cdots * \frac{-b_{j_1 \mp 2}}{a_{j_1 \mp 2}} * \frac{-b_{j_1 \mp 1}}{a_{j_1 \mp 1}}) \circ (-h_1, \mp 1, -h_3, 0)$ and $N(\frac{z_1}{v_1} + \cdots + \frac{z_t}{v_t} + e_2) = N(\frac{a_{j_1}}{b_{j_1}} + \frac{a_{j_1 \pm 1}}{b_{j_1 \pm 1}} + \cdots + \frac{a_{j_1 \mp 2}}{b_{j_1 \mp 2}} + \frac{a_{j_1 \mp 1}}{b_{j_1 \mp 1}} +$*

$$\frac{y(1 \pm h_1) - x(h_1 + h_3 \pm h_1 h_3)}{\pm y - x(1 \pm h_3)}) \text{ where } h_1 = e_1 \mp 1.$$

Note that if $h_2 = 0$ in Case 2, the solution is the same as the solution in Case 1. We can consider only when m is odd since $T \circ (h_1, \cdots, h_{2k}) = T \circ (0, h_1, \cdots, h_{2k})$ for a tangle T.

Proof. If U is rational, $N(U)$ is a 4-plat but $N(\frac{a_1}{b_1} + \cdots + \frac{a_s}{b_s} + e_1)$ is a Montesinos link/knot by equation (1). It contradicts. So U is not rational. Since U is a generalized M-tangle which is not rational, it can be written as $U = (\frac{c_1}{d_1} + \cdots + \frac{c_n}{d_n}) \circ (h_1, \cdots, h_m)$ where c_i, d_i and h_j are integers such that $0 < c_i < d_i$ for $1 \le i \le n$, $1 \le j \le m$ and $n \ge 2$. W.L.O.G, we can assume m is odd.
Case 1: If $m = 1$, from equation (1),

$$\begin{aligned} N(U + \frac{0}{1}) =& N((\frac{c_1}{d_1} + \cdots + \frac{c_n}{d_n}) \circ (h_1) + (0)) \\ =& N(\frac{c_1}{d_1} + \cdots + \frac{c_n}{d_n} + h_1) \\ =& N(\frac{a_1}{b_1} + \cdots + \frac{a_s}{b_s} + e_1). \end{aligned}$$

Then by theorem 2.1, $h_1 = e_1$, $\frac{c_i}{d_i} = \frac{a_j}{b_j}$ with $n = s$ where $i = 1, \cdots, n$, $j = j_1, j_1 + 1, \cdots, s, 1, 2, \cdots, j_1 - 2, j_1 - 1$ or $j = j_1, j_1 - 1, \cdots, 2, 1, s, \cdots, j_1 + 2, j_1 + 1$ for some $1 \le j_1 \le s$. Here, $n \ge 3$ since $s \ge 3$.
From equation (2),

$$\begin{aligned} N(U + \frac{x}{y}) =& N((\frac{c_1}{d_1} + \cdots + \frac{c_n}{d_n}) \circ (h_1) + \frac{x}{y}) \\ =& N((\frac{c_1}{d_1} + \cdots + \frac{c_n}{d_n}) + \frac{x}{y} \circ (h_1)) \quad \text{by lemma 3.3} \\ =& N(\frac{c_1}{d_1} + \cdots + \frac{c_n}{d_n} + \frac{x}{y} + h_1) \\ =& N(\frac{z_1}{v_1} + \cdots + \frac{z_t}{v_t} + e_2). \end{aligned}$$

Then by the above result, $N(\frac{z_1}{v_1} + \cdots + \frac{z_t}{v_t} + e_2) = N(\frac{a_{j_1}}{b_{j_1}} + \frac{a_{j_1 \pm 1}}{b_{j_1 \pm 1}} + \cdots + \frac{a_{j_1 \mp 2}}{b_{j_1 \mp 2}} + \frac{a_{j_1 \mp 1}}{b_{j_1 \mp 1}} + \frac{x}{y} + e_1)$. By theorem 2.1, $e_2 = \frac{x}{y} + e_1$ and $t = s$ if $\frac{x}{y}$ is an integer and $\frac{z_k}{v_k} + e_2 = \frac{x}{y} + e_1$ for some k and $t = s + 1$ if $\frac{x}{y}$ is not an integer.

Case 2: Assume that $m = 3$ and $h_m = 0$. Then by equation(1),

$$\begin{aligned} N(U+\frac{0}{1}) =& N((\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n})\circ(h_1,h_2,0)+(0)) \\ =& N((\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n})\circ(h_1)) \\ =& N(\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n}+h_1) \\ =& N(\frac{a_1}{b_1}+\cdots+\frac{a_s}{b_s}+e_1). \end{aligned}$$

Then by theorem 2.1, $h_1 = e_1$, $\frac{c_i}{d_i} = \frac{a_j}{b_j}$ with $n = s$ where $i = 1, \cdots, n$, $j = j_1, j_1+1, \cdots, s, 1, 2, \cdots, j_1-2, j_1-1$ or $j = j_1, j_1-1, \cdots, 2, 1, s, \cdots, j_1+2, j_1+1$ for some $1 \leq j_1 \leq s$. Here, $n \geq 3$.
From equation (2),

$$\begin{aligned} N(U+\frac{x}{y}) =& N((\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n})\circ(h_1,h_2,0)+\frac{x}{y}) \\ =& N((\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n})+(\frac{x}{y})\circ(0,h_2,h_1)) \quad \text{by lemma 3.3} \\ =& N((\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n})+(\frac{x}{y})\circ(0,h_2,0)+h_1) \\ =& N((\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n})+\frac{1}{h_2+\frac{1}{\frac{x}{y}}}+h_1) \\ =& N(\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n}+\frac{x}{h_2x+y}+h_1) \\ =& N(\frac{z_1}{v_1}+\cdots+\frac{z_t}{v_t}+e_2). \end{aligned}$$

Then by the above result, $N(\frac{z_1}{v_1}+\cdots+\frac{z_t}{v_t}+e_2) = N(\frac{a_{j_1}}{b_{j_1}}+\frac{a_{j_1\pm1}}{b_{j_1\pm1}}+\cdots+\frac{a_{j_1\mp2}}{b_{j_1\mp2}}+\frac{a_{j_1\mp1}}{b_{j_1\mp1}}+\frac{x}{h_2x+y}+e_1)$. By theorem 2.1, $\frac{x}{h_2x+y}+e_1 = e_2$ and $t = s$ if $\frac{x}{h_2x+y}$ is an integer and $\frac{x}{h_2x+y}+e_1 = \frac{z_k}{v_k}+e_2$ for some k and $t = s+1$ if $\frac{x}{h_2x+y}$ is not an integer.

Case 3: Assume that $m > 3$ or $m = 3$ and $h_m \neq 0$. If we assume that h_j's have the same sign and $h_j \neq 0$ for $2 \leq j \leq m-1$, then by theorem 3.9, $U = (\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n})\circ(h_1,\cdots,h_m)$ or $(\frac{-d_1}{c_1}*\cdots*\frac{-d_n}{c_n})\circ(-h_1,\cdots,-h_m,0)$. Since $N(U)$ is a Montesinos link by equation (1), case (3) in theorem 3.9 is ruled out.
(1) If $U = (\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n})\circ(h_1,\cdots,h_m)$ where $0 < c_i < d_i$, m is odd, h_j's have the same sign, $h_j \neq 0$ for $2 \leq j \leq m-1$, then by equation (1),

$$
\begin{aligned}
N(U+\frac{0}{1}) =& N((\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n})\circ(h_1,\cdots,h_m)+(0)) \\
=& N((\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n})+(0)\circ(h_m,\cdots,h_1)) \quad \text{by lemma 3.3} \\
=& N((\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n})+(h_m,\cdots,h_1)) \\
=& N((\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n})+h_1+\frac{1}{h_2+\cdots+\frac{1}{h_m}}) \\
=& N(\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n}+\frac{E[h_1,\cdots,h_m]}{E[h_2,\cdots,h_m]}) \quad \text{by proposition ??} \\
=& N(\frac{a_1}{b_1}+\cdots+\frac{a_s}{b_s}+e_1).
\end{aligned}
$$

Since h_j's have the same sign, m is odd, $m \geq 3$ and $h_m \neq 0$ if $m = 3$, $\frac{E[h_1,\cdots,h_m]}{E[h_2,\cdots,h_m]}$ cannot be an integer. Then by theorem 2.1, $\frac{c_i}{d_i} = \frac{a_j}{b_j}$ with $n = s-1$ where $i = 1,\cdots,n$, $j = j_1, j_1+1,\cdots,s,1,2,\cdots,j_1-2$ or $j = j_1, j_1-1,\cdots,2,1,s,\cdots,j_1+2$ and $\frac{E[h_1,\cdots,h_m]}{E[h_2,\cdots,h_m]} = \frac{a_{j_1\mp1}}{b_{j_1\mp1}} + e_1$ for some $1 \leq j_1 \leq s$. Here, $n \geq 2$.
From equation (2),

$$
\begin{aligned}
N(U+\frac{x}{y}) =& N((\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n})\circ(h_1,\cdots,h_m)+(l_1,\cdots,l_k)) \\
& \text{where } \frac{x}{y}=(l_1,\cdots,l_k). \\
=& N((\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n})+(l_1,\cdots,l_k)\circ(h_m,\cdots,h_1)) \quad \text{by lemma 3.3} \\
=& N(\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n}+(l_1,\cdots,l_k+h_m,\cdots,h_1)) \quad \text{by lemma 3.4} \\
=& N(\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n}+[h_1,h_2,\cdots,h_m+l_k,\cdots,l_2,l_1]) \\
=& N(\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n}+\frac{xE[h_1,\cdots,h_{m-1}]+yE[h_1,\cdots,h_m]}{xE[h_2,\cdots,h_{m-1}]+yE[h_2,\cdots,h_m]}) \\
& \text{by proposition ?? and lemma 3.5} \\
=& N(\frac{z_1}{v_1}+\cdots+\frac{z_t}{v_t}+e_2).
\end{aligned}
$$

Then by the above result, $N(\frac{z_1}{v_1}+\cdots+\frac{z_t}{v_t}+e_2) = N(\frac{a_{j_1}}{b_{j_1}}+\frac{a_{j_1\pm1}}{b_{j_1\pm1}}+\cdots+\frac{a_{j_1\mp3}}{b_{j_1\mp3}}+\frac{a_{j_1\mp2}}{b_{j_1\mp2}}+\frac{xE[h_1,\cdots,h_{m-1}]+yE[h_1,\cdots,h_m]}{xE[h_2,\cdots,h_{m-1}]+yE[h_2,\cdots,h_m]})$ where $\frac{E[h_1,\cdots,h_m]}{E[h_2,\cdots,h_m]} = \frac{a_{j_1\mp1}}{b_{j_1\mp1}}+e_1$ for some $1 \leq j_1 \leq s$.
(2) If $U = (\frac{-d_1}{c_1}*\cdots*\frac{-d_n}{c_n})\circ(-h_1,\cdots,-h_m,0)$ where $0 < c_i < d_i$, m is odd, h_j's

have the same sign, $h_j \neq 0$ for $2 \leq j \leq m-1$, then by equation (1),

$$\begin{aligned}N(U+\frac{0}{1}) =& N((\frac{-d_1}{c_1} * \cdots * \frac{-d_n}{c_n}) \circ (-h_1, \cdots, -h_m, 0)) \\ =& D((\frac{c_1}{d_1} + \cdots + \frac{c_n}{d_n}) \circ (h_1, \cdots, h_m)) \\ =& N((\frac{c_1}{d_1} + \cdots + \frac{c_n}{d_n}) \circ (h_1, \cdots, h_m) + (0,0)) \\ =& N(\frac{c_1}{d_1} + \cdots + \frac{c_n}{d_n} + (0,0) \circ (h_m, \cdots, h_1)) \quad \text{by lemma 3.3}\end{aligned}$$

$$\begin{aligned} =& N(\frac{c_1}{d_1} + \cdots + \frac{c_n}{d_n} + (h_{m-1}, \cdots, h_1)) \\ =& N(\frac{c_1}{d_1} + \cdots + \frac{c_n}{d_n} + h_1 + \frac{1}{h_2 + \cdots + \frac{1}{h_{m-1}}}) \\ =& N(\frac{c_1}{d_1} + \cdots + \frac{c_n}{d_n} + \frac{E[h_1, \cdots, h_{m-1}]}{E[h_2, \cdots, h_{m-1}]}) \quad \text{by proposition ??} \\ =& N(\frac{a_1}{b_1} + \cdots + \frac{a_s}{b_s} + e_1).\end{aligned}$$

If $m > 3$ or $m = 3$ and $h_2 \neq \pm 1$, $\dfrac{E[h_1, \cdots, h_{m-1}]}{E[h_2, \cdots, h_{m-1}]}$ cannot be an integer since h_j's have the same sign. Then by theorem 2.1, $\dfrac{c_i}{d_i} = \dfrac{a_j}{b_j}$ with $n = s-1$ where $i = 1, \cdots, n$, $j = j_1, j_1+1, \cdots, s, 1, 2, \cdots, j_1 - 2$ or $j = j_1, j_1 - 1, \cdots, 2, 1, s, \cdots, j_1 + 2$ and $\dfrac{E[h_1, \cdots, h_{m-1}]}{E[h_2, \cdots, h_{m-1}]} = \dfrac{a_{j_1 \mp 1}}{b_{j_1 \mp 1}} + e_1$ for some $1 \leq j_1 \leq s$. Here, $n \geq 2$. If $m = 3$ and $h_2 = \pm 1$, then $\dfrac{E[h_1, h_2]}{E[h_2]} = \dfrac{h_1 h_2 + 1}{h_2} = h_1 \pm 1$. By theorem 2.1, $\dfrac{c_i}{d_i} = \dfrac{a_j}{b_j}$ with $n = s$ where $i = 1, \cdots, n$, $j = j_1, j_1+1, \cdots, s, 1, 2, \cdots, j_1 - 1$ or $j = j_1, j_1 - 1, \cdots, 2, 1, s, \cdots, j_1 + 1$ and $h_1 \pm 1 = e_1$.

From equation (2),

$$N(U + \frac{x}{y}) = N((\frac{-d_1}{c_1} * \cdots * \frac{-d_n}{c_n}) \circ (-h_1, \cdots, -h_m, 0) + (l_1, \cdots, l_k))$$

where $\dfrac{x}{y} = (l_1, \cdots, l_k)$ and k is odd,

$$\begin{aligned} =& N((\frac{-d_1}{c_1} * \cdots * \frac{-d_n}{c_n}) \circ (-h_1, \cdots, -h_m, 0) \circ (l_k, \cdots, l_1)) \quad \text{by lemma 3.3} \\ =& N((\frac{-d_1}{c_1} * \cdots * \frac{-d_n}{c_n}) \circ (-h_1, \cdots, -h_m, l_k, \cdots, l_1)) \quad \text{by lemma 3.4} \\ =& D((\frac{c_1}{d_1} + \cdots + \frac{c_n}{d_n}) \circ (h_1, \cdots, h_m, -l_k, \cdots, -l_1, 0))\end{aligned}$$

$=N((\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n})\circ(h_1,\cdots,h_m,-l_k,\cdots,-l_1,0)+(0,0))$
$=N(\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n}+(0,0)\circ(0,-l_1,\cdots,-l_k,h_m,\cdots,h_1))$ by lemma 3.3
$=N(\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n}+(-l_1,\cdots,-l_k,h_m,\cdots,h_1))$
$=N(\frac{c_1}{d_1}+\cdot+\frac{c_n}{d_n}+\frac{E[h_1,\cdots,h_m,0]E[-l_1,\cdots,-l_{k-1}]+E[h_1,\cdots,h_m]E[-l_1,\cdots,-l_k]}{E[h_2,\cdots,h_m,0]E[-l_1,\cdots,-l_{k-1}]+E[h_2,\cdots,h_m]E[-l_1,\cdots,-l_k]})$
$=N(\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n}+\frac{yE[h_1,\cdots,h_m,0]-xE[h_1,\cdots,h_m]}{yE[h_2,\cdots,h_m,0]-xE[h_2,\cdots,h_m]})$
$=N(\frac{c_1}{d_1}+\cdots+\frac{c_n}{d_n}+\frac{yE[h_1,\cdots,h_{m-1}]-xE[h_1,\cdots,h_m]}{yE[h_2,\cdots,h_{m-1}]-xE[h_2,\cdots,h_m]})$
by proposition **??** and lemma 3.5
$=N(\frac{z_1}{v_1}+\cdots+\frac{z_t}{v_t}+e_2)$.

By the above result, if $m>3$ or $m=3$ and $h_2\neq\pm1$, then $N(\frac{z_1}{v_1}+\cdots+\frac{z_t}{v_t}+e_2)=N(\frac{a_{j_1}}{b_{j_1}}+\frac{a_{j_1\pm1}}{b_{j_1\pm1}}+\cdots+\frac{a_{j_1\mp3}}{b_{j_1\mp3}}+\frac{a_{j_1\mp2}}{b_{j_1\mp2}}+\frac{yE[h_1,\cdots,h_{m-1}]-xE[h_1,\cdots,h_m]}{yE[h_2,\cdots,h_{m-1}]-xE[h_2,\cdots,h_m]})$ where $\frac{E[h_1,\cdots,h_{m-1}]}{E[h_2,\cdots,h_{m-1}]}=\frac{a_{j_1\mp1}}{b_{j_1\mp1}}+e_1$ for some $1\leq j_1\leq s$. If $m=3$ and $h_2=\pm1$, then $N(\frac{z_1}{v_1}+\cdots+\frac{z_t}{v_t}+e_2)=N(\frac{a_{j_1}}{b_{j_1}}+\frac{a_{j_1\pm1}}{b_{j_1\pm1}}+\cdots+\frac{a_{j_1\mp2}}{b_{j_1\mp2}}+\frac{a_{j_1\mp1}}{b_{j_1\mp1}}+\frac{y(1\pm h_1)-x(h_1+h_3\pm h_1h_3)}{\pm y-x(1\pm h_3)})$ where $h_1=e_1\mp1$. □

Next, the system of tangle equations is solved when $s\leq2$ and $t\geq3$. If $s\leq2$, then the righthand side of equation (1) in theorem 5.1 is a rational link. So it can be written as $N(\frac{a}{b})$.

Theorem 5.2. *Suppose that a,b,x,y are integers. For $t\geq3$,*

$$N(U+\frac{0}{1})=N(\frac{a}{b})\quad(1)$$
$$\text{and } N(U+\frac{x}{y})=N(\frac{z_1}{v_1}+\cdots+\frac{z_t}{v_t}+e_2)\quad(2)$$
$$\text{where } z_j,v_j,e_2 \text{ are integers and } 0<z_j<v_j \text{ for } 1\leq j\leq t$$
$$\text{and } U \text{ is a generalized M-tangle.}$$

if and only if $t=3$,
$U=(\frac{c_1}{d_1}+\frac{pa-c_1b}{d_1b-qa})\circ(h,0)$ *and* $(\frac{pa-c_1b}{d_1b-qa}+\frac{c_1}{d_1})\circ(h,0)$ *for all integers c_1,d_1,p*

and q such that $d_1p - qc_1 = 1$ and $0 < c_1 < d_1$ where $(\frac{c_1}{d_1}, \frac{pa - c_1b}{d_1b - qa}, \frac{x}{hx + y}) = (\frac{z_{i_1}}{v_{i_1}}, \frac{z_{i_2}}{v_{i_2}} + k, \frac{z_{i_3}}{v_{i_3}} + e_2 - k)$ for some integer k where $\{i_1, i_2, i_3\}$ is cyclic permutations of $(1, 2, 3)$ and reversal of order.
Note that the choice of c_1 and p such that $d_1p - qc_1 = 1$ has no effect on U.

Proof. If U is rational, $N(U + \frac{x}{y})$ is a rational link but $N(\frac{z_1}{v_1} + \cdots + \frac{z_t}{v_t} + e_2)$ is a Montesinos link/knot by the equation (2). It contradicts. So U should be a generalized M-tangle which is not rational. Since $N(U) = N(\frac{a}{b})$ is rational by equation (1), $U = (\frac{c_1}{d_1} + \frac{c_2}{d_2}) \circ (h, 0)$ where $0 < c_i < d_i$ for $i = 1, 2$. From the proof of theorem 3 in [4], $U = (\frac{c_1}{d_1} + \frac{pa - c_1b}{d_1b - qa}) \circ (h, 0)$ and $(\frac{pa - c_1b}{d_1b - qa} + \frac{c_1}{d_1}) \circ (h, 0)$ for integers c_1, d_1, p and q such that $d_1p - qc_1 = 1$. Note that if d_1 and q are specified, then the choice of c_1 and p such that $d_1p - qc_1 = 1$ has no effect on U since $\frac{c_1}{d_1} + \frac{pa - c_1b}{d_1b - qa} = \frac{c_1 + d_1i}{d_1} + \frac{pa - c_1b - (d_1b - qa)i}{d_1b - qa} = \frac{c_1 + d_1i}{d_1} + \frac{(p + qi)a - (c_1 + d_1i)b}{d_1b - qa}$ and $d_1(p + qi) - q(c_1 + d_1i) = 1$ if and only if $d_1p - qc_1 = 1$ [4].
From equation (2),

$$\begin{aligned}
N(U + \frac{x}{y}) &= N((\frac{c_1}{d_1} + \frac{c_2}{d_2}) \circ (h, 0) + \frac{x}{y}) \\
&= N((\frac{c_1}{d_1} + \frac{c_2}{d_2}) + (\frac{x}{y}) \circ (h, 0)) \\
&= N(\frac{c_1}{d_1} + \frac{c_2}{d_2} + \frac{x}{hx + y}) \\
&= N(\frac{z_1}{v_1} + \cdots + \frac{z_t}{v_t} + e_2).
\end{aligned}$$

Since $t \geq 3$, $\frac{x}{hx + y}$ cannot be an integer. Actually, $t = 3$ and $N(\frac{z_1}{v_1} + \frac{z_2}{v_2} + \frac{z_3}{v_3} + e_2) = N(\frac{c_1}{d_1} + \frac{pa - c_1b}{d_1b - qa} + \frac{x}{hx + y})$. Similarly, for $U = (\frac{pa - c_1b}{d_1b - qa} + \frac{c_1}{d_1}) \circ (h, 0)$, $N(\frac{z_1}{v_1} + \frac{z_2}{v_2} + \frac{z_3}{v_3} + e_2) = N(\frac{pa - c_1b}{d_1b - qa} + \frac{c_1}{d_1} + \frac{x}{hx + y})$ by equation (2). Note that by theorem 2.1, $N(\frac{pa - c_1b}{d_1b - qa} + \frac{c_1}{d_1} + \frac{x}{hx + y}) = N(\frac{c_1}{d_1} + \frac{pa - c_1b}{d_1b - qa} + \frac{x}{hx + y})$. Thus $(\frac{c_1}{d_1}, \frac{pa - c_1b}{d_1b - qa}, \frac{x}{hx + y}) = (\frac{z_{i_1}}{v_{i_1}}, \frac{z_{i_2}}{v_{i_2}} + k, \frac{z_{i_3}}{v_{i_3}} + e_2 - k)$ for some integer k where $\{i_1, i_2, i_3\}$ is cyclic permutations of $(1, 2, 3)$ and reversal of order by theorem 2.1.□

Next, the system of tangle equations is solved when $s \geq 3$ and $t \leq 2$. If $t \leq 2$, then the righthand side of equation (2) in theorem 5.1 is a rational link. So it can be written as $N(\frac{z}{v})$.

Theorem 5.3. *Suppose that x, y, z, v are integers. For $s \geq 3$,*

$$\begin{aligned} N(U + \frac{0}{1}) =& N(\frac{a_1}{b_1} + \cdots + \frac{a_s}{b_s} + e_1) \quad (1) \\ \textit{and } N(U + \frac{x}{y}) =& N(\frac{z}{v}) \quad (2) \end{aligned}$$

where a_i, b_i, e_1 are integers and $0 < a_i < b_i$ for $1 \leq i \leq s$,
and U is a generalized M-tangle.

if and only if $s = 3$,
$U = (\frac{c_1}{d_1} + \frac{pz - c_1 v}{d_1 v - qz}) \circ (h, 0) \circ (-\frac{x}{y})$ *and* $(\frac{pz - c_1 v}{d_1 v - qz} + \frac{c_1}{d_1}) \circ (h, 0) \circ (-\frac{x}{y})$ *for integers c_1, d_1, p and q such that $d_1 p - qc_1 = 1$ and $0 < c_1 < d_1$ where* $(\frac{c_1}{d_1}, \frac{pz - c_1 v}{d_1 v - qz}, \frac{x}{hx - y'}) = (\frac{a_{i_1}}{b_{i_1}}, \frac{a_{i_2}}{b_{i_2}} + k, \frac{a_{i_3}}{b_{i_3}} + e_1 - k)$ *for some integer k where $\{i_1, i_2, i_3\}$ is cyclic permutations of $(1, 2, 3)$ and reversal of order and y' such that $yy'^{\pm 1} = 1$ mod x.*
Note that the choice of c_1 and p such that $d_1 p - qc_1 = 1$ has no effect on U.

Proof. If U is rational, then $N(U)$ is a rational link but by equation (1), $N(\frac{a_1}{b_1} + \cdots + \frac{a_s}{b_s} + e_1)$ is a Montesinos knot. It contradicts. So U is a nonrational generalized M-tangle.
By equation (2),

$$\begin{aligned} N(U + \frac{x}{y}) =& N(U + (l_1, \cdots, l_k)) \text{ where } \frac{x}{y} = (l_1, \cdots, l_k) \text{ and } k \text{ is odd} \\ =& N(U \circ (l_k, \cdots, l_1) + (\frac{0}{1})) \text{ by lemma 3.3} \\ =& N(\frac{z}{v}) \end{aligned}$$

Since U is a nonrational generalized M-tangle, so is $U \circ (l_k, \cdots, l_1)$. Since $N(U \circ (l_k, \cdots, l_1)) = N(\frac{z}{v})$ is rational, we can write $U \circ (l_k, \cdots, l_1) = (\frac{c_1}{d_1} + \frac{c_2}{d_2}) \circ (h, 0)$ where $0 < c_i < d_i$ for $i = 1, 2$. From the proof of theorem 3 in [4], $U \circ (l_k, \cdots, l_1) = (\frac{c_1}{d_1} + \frac{pz - c_1 v}{d_1 v - qz}) \circ (h, 0)$ and $(\frac{pz - c_1 v}{d_1 v - qz} + \frac{c_1}{d_1}) \circ (h, 0)$ for integers c_1, d_1, p and q such that $d_1 p - qc_1 = 1$. Thus $U = (\frac{c_1}{d_1} + \frac{pz - c_1 v}{d_1 v - qz}) \circ (h, 0) \circ (-l_1, \cdots, -l_k) = (\frac{c_1}{d_1} + \frac{pz - c_1 v}{d_1 v - qz}) \circ (h, 0) \circ (-\frac{x}{y})$ and $U = (\frac{pz - c_1 v}{d_1 v - qz} + \frac{c_1}{d_1}) \circ (h, 0) \circ (-\frac{x}{y})$ for integers c_1, d_1, p and q satisfying the above. Note that if d_1 and q are specified, then the choice of c_1 and p such that $d_1 p - qc_1 = 1$ has no effect on U.

By equation(1),

$$\begin{aligned}
N(U+\frac{0}{1}) =& N((\frac{c_1}{d_1}+\frac{pz-c_1v}{d_1v-qz})\circ(h,0)\circ(-l_1,\cdots,-l_k)+(0)) \\
=& N((\frac{c_1}{d_1}+\frac{pz-c_1v}{d_1v-qz})\circ(h,0)+(-l_k,\cdots,-l_1)) \text{ by lemma 3.3} \\
=& N((\frac{c_1}{d_1}+\frac{pz-c_1v}{d_1v-qz})\circ(h,0)+(-\frac{x}{y'})) \text{ where } yy'^{\pm 1}=1 \bmod x \\
=& N((\frac{c_1}{d_1}+\frac{pz-c_1v}{d_1v-qz})+(-\frac{x}{y'})\circ(h,0)) \\
=& N(\frac{c_1}{d_1}+\frac{pz-c_1v}{d_1v-qz}+\frac{x}{hx-y'}) \\
=& N(\frac{a_1}{b_1}+\cdots+\frac{a_s}{b_s}+e_1)
\end{aligned}$$

Since $s\geq 3$, $\frac{x}{hx-y'}$ cannot be an integer. Actually, $s=3$ and $N(\frac{a_1}{b_1}+\frac{a_2}{b_2}+\frac{a_3}{b_3}+e_1)=N(\frac{c_1}{d_1}+\frac{pz-c_1v}{d_1v-qz}+\frac{x}{hx-y'})$. Similarly, for $U=(\frac{pz-c_1v}{d_1v-qz}+\frac{c_1}{d_1})\circ(h,0)$, $N(\frac{a_1}{b_1}+\frac{a_2}{b_2}+\frac{a_3}{b_3}+e_1)=N(\frac{pz-c_1v}{d_1v-qz}+\frac{c_1}{d_1}+\frac{x}{hx-y'})$. Note that by theorem 2.1, $N(\frac{pz-c_1v}{d_1v-qz}+\frac{c_1}{d_1}+\frac{x}{hx-y'})=N(\frac{c_1}{d_1}+\frac{pz-c_1v}{d_1v-qz}+\frac{x}{hx-y'})$. Thus $(\frac{c_1}{d_1},\frac{pz-c_1v}{d_1v-qz},\frac{x}{hx-y'})=(\frac{a_{i_1}}{b_{i_1}},\frac{a_{i_2}}{b_{i_2}}+k,\frac{a_{i_3}}{b_{i_3}}+e_1-k)$ for some integer k where $\{i_1,i_2,i_3\}$ is cyclic permutations of $(1,2,3)$ and reversal of order by theorem 2.1.□

Example 5.4. Solve the following tangle equations where U is a generalized M-tangle.

$$N(U+\frac{0}{1})=N(\frac{1}{2}+\frac{2}{3}+\frac{2}{3}+(-3))$$
$$\text{and}\quad N(U+\frac{x}{y})=N(\frac{1}{2}+\frac{3}{5}+\frac{2}{3}+(-3)).$$

By theorem 5.1, if $m=1$, then $U=(\frac{1}{2}+\frac{2}{3}+\frac{2}{3})\circ(-3)$,$(\frac{2}{3}+\frac{2}{3}+\frac{1}{2})\circ(-3)$ or $(\frac{2}{3}+\frac{1}{2}+\frac{2}{3})\circ(-3)$. Moreover, $N(\frac{1}{2}+\frac{3}{5}+\frac{2}{3}+(-3))=N((\frac{1}{2}+\frac{2}{3}+\frac{2}{3})+\frac{x}{y}+(-3))$, $N((\frac{2}{3}+\frac{2}{3}+\frac{1}{2})+\frac{x}{y}+(-3))$ or $N((\frac{2}{3}+\frac{1}{2}+\frac{2}{3})+\frac{x}{y}+(-3))$. By theorem 2.1, there is no $\frac{x}{y}$ satisfying the above.

Similarly, if $m = 3$ and $h_m = 0$, then by theorem 5.1, $U = (\frac{1}{2} + \frac{2}{3} + \frac{2}{3}) \circ (-3, h_2, 0)$,$(\frac{2}{3} + \frac{2}{3} + \frac{1}{2}) \circ (-3, h_2, 0)$ or $(\frac{2}{3} + \frac{1}{2} + \frac{2}{3}) \circ (-3, h_2, 0)$. Moreover, $N(\frac{1}{2} + \frac{3}{5} + \frac{2}{3} + (-3)) = N((\frac{1}{2} + \frac{2}{3} + \frac{2}{3}) + \frac{x}{h_2x + y} + (-3))$, $N((\frac{2}{3} + \frac{2}{3} + \frac{1}{2}) + \frac{x}{h_2x + y} + (-3))$ or $N((\frac{2}{3} + \frac{1}{2} + \frac{2}{3}) + \frac{x}{h_2x + y} + (-3))$ which are impossible by theorem 2.1.
Thus $m > 3$ or $m = 3$ and $h_m \neq 0$.

(1) By theorem 5.1 (3a), $U = (\frac{1}{2} + \frac{2}{3}) \circ (h_1, \cdots, h_m)$ or $(\frac{2}{3} + \frac{1}{2}) \circ (h_1, \cdots, h_m)$where $\frac{E[h_1, \cdots, h_m]}{E[h_2, \cdots, h_m]} = \frac{2}{3} + (-3) = \frac{-7}{3} = -2 + \frac{1}{-2 + \frac{1}{-1}}$. This implies that $m = 3$ and $(h_1, h_2, h_3) = (-2, -2, -1)$. Moreover, $N(\frac{1}{2} + \frac{3}{5} + \frac{2}{3} + (-3)) = N(\frac{1}{2} + \frac{2}{3} + \frac{xE[-2, -2] + yE[-2, -2, -1]}{xE[-2] + yE[-2, -1]}) = N(\frac{1}{2} + \frac{2}{3} + \frac{5x - 7y}{-2x + 3y})$. By theorem 2.1, $\frac{5x - 7y}{-2x + 3y} = \frac{3}{5} - 3 = \frac{-12}{5}$ which implies $\frac{x}{y} = -1$. Thus $U = (\frac{1}{2} + \frac{2}{3}) \circ (-2, -2, -1)$, $(\frac{2}{3} + \frac{1}{2}) \circ (-2, -2, -1)$ and $\frac{x}{y} = -1$.

(2) By theorem 5.1 (3b), $U = (\frac{-2}{1} * \frac{-3}{2}) \circ (-h_1, \cdots, -h_m, 0)$ or $(\frac{-3}{2} * \frac{-2}{1}) \circ (-h_1, \cdots, -h_m, 0)$ where $\frac{E[h_1, \cdots, h_{m-1}]}{E[h_2, \cdots, h_{m-1}]} = \frac{2}{3} + (-3) = \frac{-7}{3} = -2 + \frac{1}{-3} = (-3, -2)$. Thus $(h_{m-1}, \cdots, h_1) = (-3, -2)$. This implies that $m = 3$ and $(h_1, h_2, h_3) = (-2, -3, h_3)$ for $h_3 \leq 0$. Moreover, $N(\frac{1}{2} + \frac{3}{5} + \frac{2}{3} + (-3)) = N(\frac{1}{2} + \frac{2}{3} + \frac{yE[h_1, h_2] - xE[h_1, h_2, h_3]}{yE[h_2] - xE[h_2, h_3]}) = N(\frac{1}{2} + \frac{2}{3} + \frac{7y - x(7h_3 - 2)}{-3y - x(1 - 3h_3)})$. By theorem 2.1, $\frac{7y - x(7h_3 - 2)}{-3y - x(1 - 3h_3)} = \frac{3}{5} - 3 = \frac{-12}{5}$. Then $\frac{x}{y} = \frac{1}{h_3 - 2} = (1, h_3 - 3, 0)$ or $(-1, h_3 - 1, 0)$. Hence, $U = (\frac{-2}{1} * \frac{-3}{2}) \circ (2, 3, -h_3, 0)$, $(\frac{-3}{2} * \frac{-2}{1}) \circ (2, 3, -h_3, 0)$ and $\frac{x}{y} = \frac{1}{h_3 - 2} = (1, h_3 - 3, 0)$ or $(-1, h_3 - 1, 0)$ for $h_3 \leq 0$.

Example 5.5. Solve

$$N(U + \frac{0}{1}) = N(\frac{7}{5})$$
$$\text{and } N(U + \frac{x}{y}) = N(\frac{1}{2} + \frac{3}{5} + \frac{2}{3} - 3).$$

By theorem 5.2, $U = (\frac{c_1}{d_1} + \frac{7p-5c_1}{5d_1-7q}) \circ (h,0)$, $(\frac{7p-5c_1}{5d_1-7q} + \frac{c_1}{d_1}) \circ (h,0)$ and $N(\frac{c_1}{d_1} + \frac{7p-5c_1}{5d_1-7q} + \frac{x}{hx+y}) = N(\frac{1}{2} + \frac{3}{5} + \frac{2}{3} - 3)$ for integers c_1, d_1, p and q such that $d_1 p - qc_1 = 1$ and $0 < c_1 < d_1$. By theorem 2.1, $\frac{c_1}{d_1} = \frac{1}{2}, \frac{3}{5}$ or $\frac{2}{3}$.

(1) If $\frac{c_1}{d_1} = \frac{1}{2}$, then $\frac{7p-5c_1}{5d_1-7q} = \frac{7p-5}{10-7q} = \frac{2}{3} + k$ or $\frac{3}{5} + k$ for some integer k where $2p - q = 1$. Solving $10 - 7q = \pm 3$ and $2p - q = 1$ gives $p = q = 1$ and $k = 0$. However, there is no integers p, q such that $\frac{7p-5}{10-7q} = \frac{3+5k}{5}$ and $2p - q = 1$. Thus $U = (\frac{1}{2} + \frac{2}{3}) \circ (h,0)$ and $(\frac{2}{3} + \frac{1}{2}) \circ (h,0)$ and $\frac{x}{hx+y} = \frac{3}{5} - 3 - k = \frac{-12}{5}$. Thus $\frac{x}{y} = \frac{-12}{5+12h}$.

(2) If $\frac{c_1}{d_1} = \frac{3}{5}$, then $\frac{7p-5c_1}{5d_1-7q} = \frac{7p-15}{25-7q} = \frac{2}{3} + k$ or $\frac{1}{2} + k$ for some integer k where $5p - 3q = 1$. There is no such p and q.

(3) If $\frac{c_1}{d_1} = \frac{2}{3}$, then $\frac{7p-5c_1}{5d_1-7q} = \frac{7p-10}{15-7q} = \frac{1}{2} + k$ or $\frac{3}{5} + k$ for some integer k where $3p - 2q = 1$. There is no such p and q.

Hence, the solutions for this system of tangle equations are $U = (\frac{1}{2} + \frac{2}{3}) \circ (h,0)$ and $(\frac{2}{3} + \frac{1}{2}) \circ (h,0)$ and $\frac{x}{y} = \frac{-12}{5+12h} = (-2,-2,-2,-h,0)$ for any integer h.

Acknowledgments

This section should come before the References. Funding information may also be included here.

References

[1] F. Bonahon.Involutions et fibrés de Seifert dans les variétés de dimension 3. Thése de 3e cycle, Orsay, 1979.

[2] G. Burde and H. Zieschang. Knots. Walter de Gruyter, De Gruyter Studies in Math. 5, 1985.

[3] J.H. Conway. An enumeration of knots and links and some of their algebraic properties. *Computational Problems in Abstract Algebra (John Leech, ed.) Perg- amon Press, Oxford and New York*, 1969.

[4] I. K. Darcy. Solving unoriented tangle equations involving 4-plats. *J. Knot Theory Ramifications*, 14:993–1005, 2005

[5] C. Ernst and D. W. Sumners. A calculus for rational tangles: applications to DNA recombination. *Math. Proc. Cambridge Philos. Soc.*, 108(3):489–515, 1990.

[6] Hugo Cabrera-Ibarra. On the classification of rational 3-tangles. *J. Knot Theory Ramifications*, 12(7):921–946, 2003.

[7] Cabrera Ibarra, Hugo and Lizárraga Navarro, David A.. An algorithm based on 3-braids to solve tangle equations arising in the action of Gin DNA invertase. *Appl. Math. Comput.*, 216(1):95–106, 2010.

[8] L.H. Kauffman and S. Lambropoulou. On the classification of rational tangles. *Advances in Applied Mathematics*, 33:199–237, 2004.

[9] J. Roberts. Elementary number theory, a problem oriented approach. MIT Press, Cambridge, MA, 1977.

[10] H. Zieschang. Classification of montesinos knots. *Lect. Notes Math.*, 1060:378– 389, 1984.