Solve $\vec{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 6 & 3 \end{bmatrix} \vec{x}$

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① Find e. values:

Triangular $\Rightarrow$ e. values along diagonal

$\Rightarrow r = 1, 2, 3$

or calculate $|A - rI| = \begin{vmatrix} 1-r & 0 & 0 \\ 2 & 2-r & 0 \\ 4 & 6 & 3-r \end{vmatrix} = (1-r)(2-r)(3-r) = 0$

$\Rightarrow r = 1, 2, 3$

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② Find e. vectors

$r=1$: $A - I = \begin{bmatrix} 1-1 & 0 & 0 \\ 2 & 2-1 & 0 \\ 4 & 6 & 3-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 6 & 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 4 & 6 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{bmatrix} 2 & 0 & -\frac{1}{2} \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 is free

$2x_1 - \frac{1}{2}x_3 = 0 \Rightarrow x_1 = \frac{1}{4}x_3$

$4x_2 + 2x_3 = 0 \qquad x_2 = -\frac{1}{2}x_3$

$\qquad\qquad\qquad\qquad x_3 = x_3$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1/4 \\ -1/2 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ is an e. vector w/ e. value $r = 1$

check $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 6 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2-4 \\ 4-12+12 \end{bmatrix}$ ✓

$r=2:$ $\begin{bmatrix} 1-2 & 0 & 0 \\ 2 & 2-2 & 0 \\ 4 & 6 & 3-2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 0 & 0 \\ 4^{+4} & 6^{+0} & 1^{+0} \end{bmatrix}$

$\sim \left[\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{aligned} x_1 &= 0 \\ 6x_2 + x_3 &= 0 \\ x_3 &= x_3 \end{aligned}$

x_3 free

Check $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 6 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 6-18 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -12 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}$ ✓

$r=3:$ $\begin{bmatrix} 1-3 & 0 & 0 \\ 2 & 2-3 & 0 \\ 4 & 6 & 3-3 \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & 0 \\ 2^{+2} & -1^{+0} & 0^{+0} \\ 4 & 6 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 6 & 0 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= x_3 \end{aligned}$

Check $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 6 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ✓

$\Rightarrow$ General solns

$$\vec{X} = c_1 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} e^{t} + \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{3t}$$

Fundamental set of solns

$$\left\{ \begin{bmatrix} e^{t} \\ -2e^{t} \\ 4e^{t} \end{bmatrix}, \begin{bmatrix} 0 \\ e^{2t} \\ -6e^{2t} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix} \right\}$$

A fundamental matrix (columns store fundamental set of solns)

$$\Psi = \begin{bmatrix} e^{t} & 0 & 0 \\ -2e^{t} & e^{2t} & 0 \\ 4e^{t} & -6e^{2t} & e^{3t} \end{bmatrix}$$

Wronskian = determinant of fundamental matrix

$$W(0) = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -6 & 1 \end{vmatrix} = \underline{1}$$

To find a better basis for IVP when $t = 0$

use $\Psi(t) \cdot \Psi^{-1}(0)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Using cofactors

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -6 & 1 \end{pmatrix}^{-1}$$

$$\xrightarrow{\text{transpose}} \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\text{terms with cofactors}]{\text{replace}}$$

$$\begin{pmatrix} +\begin{vmatrix} 1 & -6 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & -6 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} -2 & 4 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} -2 & 4 \\ 1 & -6 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 0 & -6 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} \end{pmatrix} \xrightarrow{\text{calculate 2x2 determ}}$$

and divide by the 3x3 deter w(0)

$$= \frac{1}{1}\begin{pmatrix} 1 & 0 & 0 \\ +2 & 1 & 0 \\ 8 & 6 & 1 \end{pmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -6 & 1 & 0 & 0 & 1 \end{array}\right] \leftarrow AX = I$$

$R_2 + 2R_1 \to R_2$
$R_3 - 4R_1 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & -6 & 1 & -4 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 8 & 6 & 1 \end{array}\right] \qquad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 8 & 6 & 1 \end{bmatrix}}_{[\Psi(0)]^{-1}}$$

$$\begin{bmatrix} e^t & 0 & 0 \\ -2e^t & e^{2t} & 0 \\ 4e^t & -6e^{2t} & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 8 & 6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 0 & 0 \\ -2e^t + 2e^{2t} & e^{2t} & 0 \\ 4e^t - 12e^{2t} + 8e^{3t} & -6e^{2t} + 6e^{3t} & e^{3t} \end{bmatrix} = \Phi(t)$$

$\Rightarrow$ "Better" ~~soln~~ form of soln

$$\vec{x} = c_1 \begin{bmatrix} e^t \\ -2e^t + 2e^{2t} \\ 4e^t - 12e^{2t} + 8e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{2t} \\ -6e^{2t} + 6e^{3t} \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix}$$

Solve IVP: $\vec{x}(0) = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$

$$\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_3 \\ c_4 \\ c_5 \end{pmatrix}$$

$\Rightarrow c_1 = 7, \ c_2 = 8, \ c_3 = 9$

Why it works:

Solving 3 initial value problems at once

$X(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -6 & 1 \end{pmatrix}}_{\substack{\text{~~Wronsk~~ Fundamental} \\ \text{matrix = coefficient matrix}}} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -6 & 1 \end{pmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 8 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}$$

$$\Rightarrow 1^{rst} \text{ colm of } \Phi(t) = 1 \begin{pmatrix} e^t \\ -2e^t \\ 4e^t \end{pmatrix} + 2 \begin{pmatrix} 0 \\ e^{2t} \\ -6e^{2t} \end{pmatrix} + 8 \begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} e^t \\ -2e^t + 2e^{2t} \\ 4e^t - 12e^{2t} \end{pmatrix}$$

Similarly 2nd column comes from solving IVP $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
3rd " " " " $\vec{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \Phi(t) = \Psi(t) \cdot \underbrace{\Psi^{-1}(0) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}}$$