2.1 Solve first order linear DE

$$1\,y' + p(t)\,y = g(t)$$

using an integrating factor

$$u(t) = e^{\int p(t)\,dt}$$

Example: $1\,y' + \underbrace{3t^2}_{p(t)}\,y = t^2$

To use formula need 1 in front of y'

$$u(t) = e^{\int 3t^2\,dt} = e^{t^3+C}$$

only need 1 $u(t)$, take $u(t) = e^{t^3}$

To create product rule

Multiply both sides of DE by integrating factor $u(t) = e^{t^3}$

$$e^{t^3}\left[y' + 3t^2 y\right] = t^2 e^{t^3}$$

product rule

$$e^{t^3} y' + \left(e^{t^3}\right)\left(3t^2\right) y = t^2 e^{t^3}$$

check this step ✓

$$\left(e^{t^3} y\right)' = t^2 e^{t^3}$$

$$\int \left(e^{t^3} y\right)' dt = \int t^2 e^{t^3} dt$$

LHS

$$\left(e^{t^3}\right) y = \int t^2 e^{t^3} dt$$

RHS

$$\boxed{\begin{aligned} &\text{Let } u = t^3 \\ &\tfrac{1}{3} du = \tfrac{3t^2}{3} dt \end{aligned}}$$

For Integrating RHS

- integration by substitution
- " " parts
- " " partial fractions

$$= \frac{1}{3} \int e^u \, du$$

$$= \frac{1}{3} e^u + C$$

$$e^{t^3} y = \frac{1}{3} e^{t^3} + C$$

$$\underbrace{e^{t^3}}_{u(t)} y = \frac{1}{3}e^{t^3} + C$$

Solve for y:

$$\boxed{y = \frac{1}{3} + Ce^{-t^3}}$$

Ex 2: $\boxed{t^3 y' + 3t^2 y = 4}$

get 1 in front of y'

notice product rule

$$(t^3 y)' = 4$$

$$\frac{t^3 y'}{t^3} + \frac{3t^2 y}{t^3} = \frac{4}{t^3}$$

$$y' + \underbrace{3t^{-1}}_{p(t)} y = 4t^{-3}$$

Integrating factor $= u(t) = e^{\int 3t^{-1}\,dt} = e^{3\int t^{-1}\,dt}$

$$= e^{3\ln|t|} = e^{\ln|t|^3} = |t|^3$$

Let $u(t) = t^3$

Multiply both sides by $u(t) = t^3$

Multiply both sides by $u(t) = t^{-}$

$$t^3\left[y' + 3t^{-1}y\right] = \left[4t^{-3}\right]t^3$$

$$\boxed{t^3y' + 3t^2y = 4}$$

check product rule ✓

$$\int (t^3y)'\,dt = \int 4\,dt$$

$$t^3y = 4t + C$$

$$\boxed{y = 4t^{-2} + Ct^{-3}}$$