Introduction to Ordinary Differential Equations: Homework 1

Determine if the following functions are 1:1. Prove it.

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ $\boxed{f \text{ is not 1:1.}}$

   Notice that $f(-1) = 1 = f(1)$ but $-1 \neq 1$.

2. $f : [0, \infty) \to \mathbb{R}, f(x) = x^2$ $\boxed{f \text{ is 1:1.}}$

   Indeed, suppose that $f(x_1) = f(x_2)$. Then $(x_1)^2 = (x_2)^2$. Since $x_1, x_2 \geq 0$, when we square root both sides, we obtain only the positive answers. In other words, $\pm\sqrt{(x_1)^2} = +x_1$ and $\pm\sqrt{(x_2)^2} = +x_2$. Thus, we conclude that $x_1 = x_2$.

3. $f : [0, \infty) \to [0, \infty), f(x) = x^2$ $\boxed{f \text{ is 1:1.}}$

   The proof is the same as the previous problem.

4. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$ $\boxed{f \text{ is 1:1.}}$

   Indeed, suppose that $f(x_1) = f(x_2)$. Then $(x_1)^3 = (x_2)^3$. After taking the cube root of both sides, we conclude that $x_1 = x_2$.

5. $f : \mathbb{R} \to \mathbb{R}, f(x) = 2$ $\boxed{f \text{ is not 1:1.}}$

   Notice that $f(-1) = 2 = f(1)$ but $-1 \neq 1$.

6. $f : \mathbb{R} \to \mathbb{R}, f(x) = 8x + 2$ $\boxed{f \text{ is 1:1.}}$

   Suppose that $f(x_1) = f(x_2)$. Then $8x_1 + 2 = 8x_2 + 2$. After subtracting 2 from both sides, we have $8x_1 = 8x_2$. Finally, after dividing both sides by 8, we have that $x_1 = x_2$.

7. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 + 3x$ $\boxed{f \text{ is not 1:1.}}$

   Notice that $f(-3) = 0 = f(0)$, but $-3 \neq 0$.

8. $f : \mathbb{R} \to \mathbb{R}, f(x) = e^x$ $\boxed{f \text{ is 1:1.}}$

   Suppose that $f(x_1) = f(x_2)$. Then $e^{x_1} = e^{x_2}$. Taking the natural log of both sides, we conclude that $x_1 = x_2$.

9. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^4 + x^2$ $\boxed{f \text{ is not 1:1.}}$

   Notice that $f(-1) = 2 = f(1)$, but $-1 \neq 1$.

10. $f : \mathbb{R} \to \mathbb{R}, f(x) = \sin x$ $\boxed{f \text{ is not 1:1.}}$

    Notice that $f(0) = 0 = f(\pi)$, but $0 \neq \pi$.

Determine if the following functions are onto. If a function is not onto, prove it.

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ $\boxed{f \text{ is not onto.}}$

   Notice that $-1 \in \mathbb{R}$, but there does not exist an $x \in \mathbb{R}$ such that $f(x) = x^2 = -1$. (The imaginary number i would be a solution, but $i \notin \mathbb{R}$).

2. $f : [0, \infty) \to \mathbb{R}, f(x) = x^2$ $\boxed{f \text{ is not onto.}}$

   The same counterexample as the previous problem applies.

3. $f : [0, \infty) \to [0, \infty), f(x) = x^2$ $\boxed{f \text{ is onto.}}$

4. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$ $\boxed{f \text{ is onto.}}$

5. $f : \mathbb{R} \to \mathbb{R}, f(x) = 2$ $\boxed{f \text{ is not onto.}}$

   Notice that $-1 \in \mathbb{R}$, but there does not exist an $x \in \mathbb{R}$ such that $f(x) = 2 = -1$.

6. $f : \mathbb{R} \to \mathbb{R}, f(x) = 8x + 2$ $\boxed{f \text{ is onto.}}$

7. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 + 3x$ $\boxed{f \text{ is not onto.}}$

   Notice that $-3 \in \mathbb{R}$, but there does not exist an $x \in \mathbb{R}$ such that $f(x) = x^2 + 3x = -3$. Indeed, suppose that $x^2 + 3x = -3$. Then $x^2 + 3x + 3 = 0$. If we apply the quadratic formula, we obtain $x = \frac{1}{2}(-3 \pm \sqrt{-3}) \notin \mathbb{R}$.

8. $f : \mathbb{R} \to \mathbb{R}, f(x) = e^x$ $\boxed{f \text{ is not onto.}}$

   Notice that $0 \in \mathbb{R}$, but there does not exist an $x \in \mathbb{R}$ such that $f(x) = e^x = 0$. Indeed, if $e^x = 0$, then $x = \ln(0)$, which is undefined.

9. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^4 + x^2$ $\boxed{f \text{ is not onto.}}$

   Notice that $-1 \in \mathbb{R}$, but there does not exist an $x \in \mathbb{R}$ such that $f(x) = x^4 + x^2 = -1$. Indeed, suppose that $x^4 + x^2 = -1$. Then $x^4 + x^2 + 1 = 0$. If we apply the quadratic formula, we obtain $x^2 = \frac{1}{2}(-1 \pm \sqrt{-3}) \notin \mathbb{R}$.

10. $f : \mathbb{R} \to \mathbb{R}, f(x) = \sin x$ $\boxed{f \text{ is not onto.}}$

    Notice that $-2 \in \mathbb{R}$, but there does not exist an $x \in \mathbb{R}$ such that $f(x) = \sin x = -2$. Indeed, $-1 \leq \sin x \leq 1$ for all $x \in \mathbb{R}$.

Determine if the following functions are invertible. If a function is not invertible, state why and determine if you can create an invertible function by changing the co-domain.

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ $\boxed{f \text{ is not invertible.}}$

   f cannot be made invertible by changing the co-domain because its domain forces it to not be 1:1.

2. $f : [0, \infty) \to \mathbb{R}, f(x) = x^2$ $\boxed{f \text{ is not invertible.}}$

   f can be made invertible by changing the co-domain to $[0, \infty)$.

3. $f : [0, \infty) \to [0, \infty), f(x) = x^2$ $\boxed{f \text{ is invertible.}}$

4. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$ $\boxed{f \text{ is invertible.}}$

5. $f : \mathbb{R} \to \mathbb{R}, f(x) = 2$ $\boxed{f \text{ is not invertible.}}$

   f cannot be made invertible by changing the co-domain since it's a constant function and will never be 1:1.

6. $f : \mathbb{R} \to \mathbb{R}, f(x) = 8x + 2$ $\boxed{f \text{ is invertible.}}$

7. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 + 3x$ $\boxed{f \text{ is not invertible.}}$

   f cannot be made invertible by changing the co-domain since its domain forces it to not be 1:1.

8. $f : \mathbb{R} \to \mathbb{R}, f(x) = e^x$ $\boxed{f \text{ is not invertible.}}$

   f can be made invertible by changing the co-domain to $(0, \infty)$.

9. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^4 + x^2$ $\boxed{f \text{ is not invertible.}}$

   f cannot be made invertible by changing the co-domain since its domain forces it to not be 1:1.

10. $f : \mathbb{R} \to \mathbb{R}, f(x) = \sin x$ $\boxed{f \text{ is not invertible.}}$

    f cannot be made invertible by changing the co-domain since it is 2π-periodic and will never be 1:1 if its domain is $\mathbb{R}$.