22M:100 (MATH:3600:0001) Exam 2
April 20, 2016 **Show all work**

[10] 1.) Find the radius of convergence of the power series $\sum_{n=2}^{\infty} \frac{(-2)^n(x-4)^n}{n^2}$

$\frac{a_{n+1}}{a_n} = \left(\frac{(-2)^{n+1}(x-4)^{n+1}}{(n+1)^2}\right)\left(\frac{n^2}{(-2)^n(x-4)^n}\right) = \frac{(-2)(x-4)(n^2)}{n^2+2n+1}$

$lim_{n\to\infty}\frac{|(-2)(x-4)(n^2)|}{|n^2+2n+1|} = |2(x-4)| < 1.$

Thus the series converges for all x such that $|x-4| < \frac{1}{2}$.

Hence the radius of convergence is $\frac{1}{2}$.

Note, you were not asked to find all x for which the above power series converges, but we will do so anyway.

We know the series diverges for all x such that $|x-4| > \frac{1}{2}$. Thus we only need to check the endpoints of $[4-\frac{1}{2}, 4+\frac{1}{2}] = [\frac{7}{2}, \frac{9}{2}]$

For $x = \frac{7}{2}$, $\sum_{n=2}^{\infty} \frac{(-2)^n(x-4)^n}{n^2} = \sum_{n=2}^{\infty} \frac{(-2)^n(\frac{7}{2}-4)^n}{n^2} = \sum_{n=2}^{\infty} \frac{(-2)^n(-\frac{1}{2})^n}{n^2} = \sum_{n=2}^{\infty} \frac{1}{n^2}$

For $x = \frac{9}{2}$, $\sum_{n=2}^{\infty} \frac{(-2)^n(x-4)^n}{n^2} = \sum_{n=2}^{\infty} \frac{(-2)^n(\frac{9}{2}-4)^n}{n^2} = \sum_{n=2}^{\infty} \frac{(-2)^n(\frac{1}{2})^n}{n^2} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2}$

Both these series converge and thus the series $\sum_{n=2}^{\infty} \frac{(-2)^n(x-4)^n}{n^2}$ converges for all $x \in [\frac{7}{2}, \frac{9}{2}]$

Thus if $f(x) = \sum_{n=2}^{\infty} \frac{(-2)^n(x-4)^n}{n^2}$, the largest possible domain for f is $[\frac{7}{2}, \frac{9}{2}]$

2.) Circle T for True and F for false.

[3] 2a.) Suppose $f(x) = \Sigma a_n(x-4)^n$ has a radius of convergence $= r$ about the point 4. Then we can define the domain of f to be $(4-r, 4+r)$.

T

[3] 2b.) Suppose $f(x) = \Sigma a_n(x-4)^n$ has a radius of convergence $= r$ about the point 4. Then we can define the domain of f to be $(r-4, r+4)$.

F

[3] 2c.) The radius of convergence of the power series for $f(x) = \frac{x}{(x^2+9)(x+5)}$ about the point $x_0 = 2$ is at least as large as$\sqrt{13}$.

T

[3] 2d.) Let $f(x) = \frac{x}{(x^2+9)(x+5)}$. Then $\frac{x}{(x^2+9)(x+5)} = \Sigma_{n=0}^{\infty} a_n(x-2)^n$ where $a_n = \frac{f^{(n)}(2)}{n!}$ for all values of $x \in (2-\sqrt{13},\ 2+\sqrt{13})$.

T

3.) Given the differential equation $2xy'' - (1+x)y' + y = 0$,

[5] i.) Determine if $x = 0$ is an ordinary point, regular singular point or irregular singular point.

Extra stuff: $y'' - \left(\frac{1+x}{2x}\right) y' + \left(\frac{1}{2x}\right) y = 0$,

Note since the coefficients are rational functions, the only values that are singular occur when either of the denominators of the coefficients of y' and y are 0.

Setting $2x = 0$ implies $x = 0$ is the only singular point. All other points are ordinary points.

False attempt to covert equation into an Euler equation where coefficient of y'' is x^2 and coefficient of y' is αx:

$x^2y'' - \left(\frac{x(1+x)}{2x}\right) xy' + \left(\frac{x^2}{2x}\right) y = x^2y'' - \left(\frac{(1+x)}{2}\right) xy' + \left(\frac{x}{2}\right) y = 0$,

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Answer: $y'' - \left(\frac{1+x}{2x}\right) y' + \left(\frac{1}{2x}\right) y = 0$ implies $x = 0$ is a singular point since setting denominators = 0 implies $2x = 0$ implies $x = 0$.

$lim_{x\to 0}\frac{-x(1+x)}{2x} = lim_{x\to 0}\frac{-(1+x)}{2} = -\frac{1}{2}$

$lim_{x\to 0}\frac{x^2}{2x} = lim_{x\to 0}\frac{x}{2} = 0$

Since both limits are finite, $x = 0$ is a regular singular point.

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[15] ii.) Determine the indicial equation, the roots of the indicial equation, and the recurrence relation.

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ where $a_0 \neq 0$.

Then $y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$, and $y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$,

$2xy'' - (1+x)y' + y = 2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - (1+x)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r}$

$= \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r}$

$= \sum_{n=0}^{\infty}[2(n+r-1)-1](n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}[-(n+r)+1]a_n x^{n+r}$

$= \sum_{n=0}^{\infty}[2n+2r-3](n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}[-(n+r)+1]a_n x^{n+r}$

$= [2r-3](r)a_0 x^{r-1} + \sum_{n=1}^{\infty}[2n+2r-3](n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}[-(n+r)+1]a_n x^{n+r}$

$= (2r-3)(r)a_0 x^{r-1} + \sum_{n=0}^{\infty}[2(n+1)+2r-3](n+r+1)a_{n+1} x^{n+r} + \sum_{n=0}^{\infty}[-(n+r)+1]a_n x^{n+r}$

$= (2r-3)(r)a_0 x^{r-1} + \sum_{n=0}^{\infty}(2n+2r-1)(n+r+1)a_{n+1} x^{n+r} + \sum_{n=0}^{\infty}(1-n-r)a_n x^{n+r}$

$= (2r-3)(r)a_0 x^{r-1} + \sum_{n=0}^{\infty}[(2n+2r-1)(n+r+1)a_{n+1} + (1-n-r)a_n]x^{n+r} = 0$

Thus the coefficient of x^k is zero for all k.

Thus $r(2r-3)a_0 = 0$. Since $a_0 \neq 0$. The indicial question $r(2r-3) = 0$ implies $r = 0, \frac{3}{2}$

Also, $(2n+2r-1)(n+r+1)a_{n+1} + (1-n-r)a_n = 0$. Thus $a_{n+1} = \frac{(n+r-1)a_n}{(2n+2r-1)(n+r+1)}$

indicial equation: $r(2r-3) = 0$ roots of the indicial equation: $r = 0, \frac{3}{2}$

recurrence relation: $a_{n+1} = \frac{(n+r-1)a_n}{(2n+2r-1)(n+r+1)}$

[5] 4i.) Show that $\mathbf{x} = \left[\begin{pmatrix}1\\0\end{pmatrix}t + \begin{pmatrix}0\\\frac{1}{2}\end{pmatrix}\right]e^t$ is a solution to the differential equation $\mathbf{x}' = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\mathbf{x}$

$$\mathbf{x}' = \begin{pmatrix}1\\0\end{pmatrix}e^t + \left[\begin{pmatrix}1\\0\end{pmatrix}t + \begin{pmatrix}0\\\frac{1}{2}\end{pmatrix}\right]e^t = \left[\begin{pmatrix}1\\0\end{pmatrix}t + \begin{pmatrix}1\\\frac{1}{2}\end{pmatrix}\right]e^t$$

$$A\mathbf{x} = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\left[\begin{pmatrix}1\\0\end{pmatrix}t + \begin{pmatrix}0\\\frac{1}{2}\end{pmatrix}\right]e^t = \left[\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}t + \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}0\\\frac{1}{2}\end{pmatrix}\right]e^t = \left[\begin{pmatrix}1\\0\end{pmatrix}t + \begin{pmatrix}1\\\frac{1}{2}\end{pmatrix}\right]e^t$$

Thus $\mathbf{x}' = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\mathbf{x}$ for $\mathbf{x} = \left[\begin{pmatrix}1\\0\end{pmatrix}t + \begin{pmatrix}0\\\frac{1}{2}\end{pmatrix}\right]e^t$

Of course in the above, you could have equivalently used $\mathbf{x} = \begin{pmatrix}t\\\frac{1}{2}\end{pmatrix}e^t = \begin{pmatrix}te^t\\\frac{1}{2}e^t\end{pmatrix}$

[5] 4ii) Find a second solution to $\mathbf{x}' = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\mathbf{x}$

$|A - \lambda I| = \begin{vmatrix}1-\lambda & 2\\0 & 1-\lambda\end{vmatrix} = (1-\lambda)^2 = 0$. Thus $\lambda = 1$ is an eigenvalue with algebraic multiplicity 2.

Or note that $\lambda = 1$ is an eigenvalue with algebraic multiplicity 2 since the diagonal entries of a triangular matrix are eigenvalues of that matrix.

$A - (1)I = \begin{pmatrix}0 & 2\\0 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & 2\\0 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

Thus a 2nd solution is $\mathbf{x} = \begin{pmatrix}1\\0\end{pmatrix}e^t$

[5] 4iii) State the general solution to $\mathbf{x}' = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\mathbf{x}$

$$\mathbf{x}(t) = c_1\begin{pmatrix}1\\0\end{pmatrix}e^t + c_2\left[\begin{pmatrix}1\\0\end{pmatrix}t + \begin{pmatrix}0\\\frac{1}{2}\end{pmatrix}\right]e^t$$

[5] 4iv) Solve the initial value problem $\mathbf{x}' = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix}3\\4\end{pmatrix}$

$c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\\frac{1}{2}\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$ implies $c_1 = 3$ and $c_2 = 8$.

Thus the solution to the IVP is $\mathbf{x}(t) = 3\begin{pmatrix}1\\0\end{pmatrix}e^t + 8\left[\begin{pmatrix}1\\0\end{pmatrix}t + \begin{pmatrix}0\\\frac{1}{2}\end{pmatrix}\right]e^t$

or equivalently, $\mathbf{x}(t) = \begin{pmatrix}3\\4\end{pmatrix}e^t + \begin{pmatrix}8\\0\end{pmatrix}te^t$

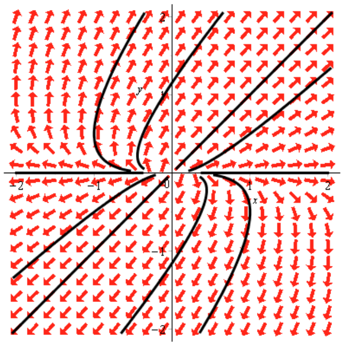

Fig. 1. Phase portrait

The phase portrait implies the corresponding differential equation $\mathbf{x}' = A\mathbf{x}$ has two real (non-repeated) positive eigenvalues. Since the eigenvalues are $\frac{p \pm \sqrt{p^2-4q}}{2}$, p, q are both positive and thus the point $(p, q) = \mathbf{E}$ in the stability diagram.

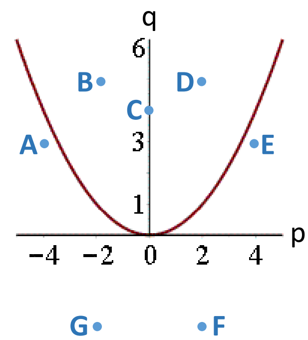

Fig 2. Stability diagram

Note the trajectory corresponding to $x_2 = 0 = (\frac{0}{1})x_1$. Thus $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of A.

Note the trajectory corresponding to $x_2 = x_1 = (\frac{1}{1})x_1$. Thus $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A.

5.) The phase portrait for a differential equation (not given) is shown above in Fig 1. Answer the following questions about this differential equation and its solution.

[9] i.) Find all equilibrium solutions and determine whether the critical point is asymptotically stable, stable, or unstable. Also classify it as to type (nodal source, nodal sink, saddle point, spiral source, spiral sink, center).

Equilibrium solution: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Stability : *unstable*

Type: *Nodal source*

[6] ii.) Which of the differential equations below matches the phase portrait shown above?

Removing all DE's where the matrix clearly does not have 2 different positive eigenvalues (note for the triangular matrices, the eigenvalues are the diagonal entries):

a.) $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \mathbf{x}$

e.) $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \mathbf{x}$

Note the matrices below do not have real eigenvalues, but that may or may not be obvious depending on whether or not you remember the special case we discussed in class.

i.) $\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \mathbf{x}$

j.) $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \mathbf{x}$

k.) $\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{x}$

To determine which of the above is the correct solution, you can do one of the following:

Method 1.) You know that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an e.vector. Calculate $A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to see that the **answer must be (a).**

Method 2.) Determine eigenvalues of the above 5 matrices to rule out i, j, k (and use this information to answer problem 6). To distinguish (a) from (c), you need to look at the eigenvectors. You can either calculate these directly, but method 1 is faster.

Note you can determine the eigenvalues using det(A - rI) or by using that p = trace of the matrix and q = determinant of the matrix,

[4] iii.) The (p, q) value corresponding to this differential equation is plotted in the Fig 2 graph above. Circle the letter corresponding to the (p, q) value corresponding to the differential equation, $\mathbf{x}' = A\mathbf{x}$ whose phase portrait is drawn above. Recall that the eigenvalues of A are $\frac{p \pm \sqrt{p^2-4q}}{2}$

E

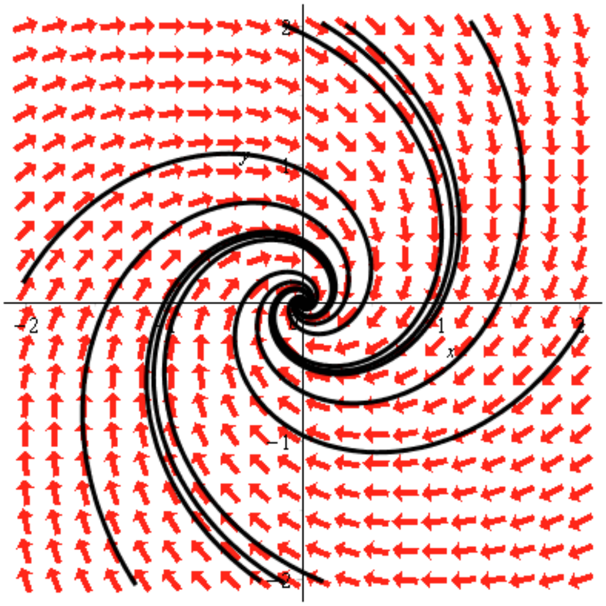

Fig. 1. Phase portrait

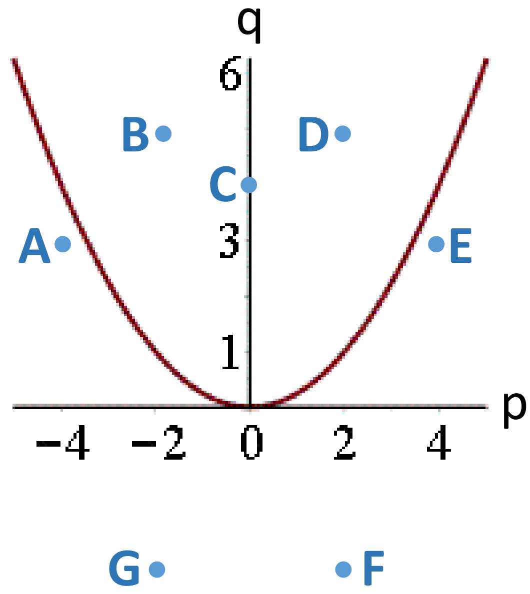

Fig 2. Stability diagram

The above phase portrait implies the corresponding differential equation $\mathbf{x}' = A\mathbf{x}$ has two complex eigenvalues, $a \pm bi$ where $a < 0$.

Since the eigenvalues are $\frac{p \pm \sqrt{p^2 - 4q}}{2}$, $p < 0$ and $p^2 - 4q < 0$ Thus $q > \frac{p}{4}$ and $p < 0$ implies the point $(p, q) =$ **B** in the stability diagram.

6.) The phase portrait for a differential equation (not given) is shown above in Fig 1. Answer the following questions about this differential equation and its solution.

[9] i.) Find all equilibrium solutions and determine whether the critical point is asymptotically stable, stable, or unstable. Also classify it as to type (nodal source, nodal sink, saddle point, spiral source, spiral sink, center).

Equilibrium solution: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Stability : *asymptotically stable* Type: *spiral sink*

[6] ii.) Which of the differential equations below matches the phase portrait shown above?

Removing all the matrices with real eigenvalues:

e.) $\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \mathbf{x}$ f.) $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \mathbf{x}$ g.) $\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{x}$

Calculate the eigenvalues of the above matrices to determine that **the answer is (g).**

[4] iii.) The (p, q) value corresponding to this differential equation is plotted in the Fig 2 graph above. Circle the letter corresponding to the (p, q) value corresponding to the differential equation, $\mathbf{x}' = A\mathbf{x}$ whose phase portrait is drawn above. Recall that the eigenvalues of A are $\frac{p \pm \sqrt{p^2 - 4q}}{2}$

B