Math 3600 Differential Equations Exam #1
March 2, 2016 SHOW ALL WORK

[20] 1.) Solve $y'' - 6y' + 9y = 0$, $y(0) = 2$, $y'(0) = 4$.

$r^2 - 6r + 9 = (r-3)^2 = 0$. Thus $r = 3$

General solution: $y = c_1 e^{3t} + c_2 t e^{3t}$

$y' = 3c_1 e^{3t} + c_2(e^{3t} + 3te^{3t})$

$y(0) = 2$: $\quad 2 = c_1$

$y'(0) = 4$: $\quad 4 = 3c_1 + c_2$. Thus $c_2 = 4 - 6 = -2$.

Answer: $\underline{y = 2e^{3t} - 2te^{3t}}$

2.) Circle T for true and F for false.

[4] 2a.) The equation $ln(t)y' = \frac{t}{t+1} - y(sint^2)$ is a linear differential equation.
T

[4] 2b.) The equation $y' + y = y^2$ is a linear differential equation. F

[4] 2c.) Suppose $y = \phi_1(t)$ and $y = \phi_2(t)$ are solutions to $ay'' + by' + cy = 0$. If $y = h(t)$ is also a solution to $ay'' + by' + cy = 0$, then there exists constants c_1 and c_2 such that $h(t) = c_1\phi_1(t) + c_2\phi_2(t)$.
F

[4] 2d.) Suppose $y = \phi_1(t)$ and $y = \phi_2(t)$ are linearly independent solutions to $ay'' + by' + cy = 0$. If $y = h(t)$ is also a solution to $ay'' + by' + cy = 0$, then there exists constants c_1 and c_2 such that $h(t) = c_1\phi_1(t) + c_2\phi_2(t)$. T

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[4] 3.) By giving a specific counter-example, prove that $y = ln(x)$ is not a linear function.

Proof 1: $ln(2) = ln(1+1) \neq 0 = ln(1) + ln(1)$.

Proof 2: $ln(e) + ln(1) = 1 + 0 = 1$. But $ln(e+1) > ln(e) = 1$ since $y = ln(x)$ is an increasing function (since $[ln(x)]' = \frac{1}{x} > 0$ for $x > 0$).

Proof 3: $ln(e) + ln(e) = 1 + 1 = 2$. But $ln(e+e) = ln(2e) = ln(2) + ln(e) = ln(2) + 1$.

$ln(2) < ln(e) = 1$. Thus $2 \neq ln(2) + 1$. Hence $ln(e) + ln(e) \neq ln(2e)$ and thus $y = ln(x)$ is not a linear function.

Proof 4: $e \cdot ln(1) = e \cdot 0 = 0 \neq 1 = ln(e) = ln(e \cdot 1)$. In other words, if $f(x) = ln(x)$, $f(cx) \neq cf(x)$ for constant $c = e$ and $x = 1$.

[20] 4.)

4a.) Circle the differential equation whose direction field is given above.

v.) $y' = \frac{y}{y+2}$

4b.) Draw the solution to the differential equation whose direction field is given above that satisfies the initial condition $y(1) = -3$

4c.) Does the differential equation whose direction field is given above have any equilibrium solutions? If so, state whether they are stable, semi-stable or unstable.

The constant solution $y = 0$ is an unstable equilibrium solution.

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[20] 5.) Solve: $y' = e^{4t} - \frac{y}{t}$

Linear, not separable: $y' + \frac{y}{t} = e^{4t}$

Let $u(t) = e^{\int \frac{dt}{t}} = e^{ln|t|+C} = C|t|$. Let $u(t) = t$

$ty' + y = te^{4t}$

$(ty)' = te^{4t}$

$\int (ty)' = \int te^{4t} dt.$

Let $u = t \qquad dv = e^{4t}$

$du = dt \qquad v = \frac{e^{4t}}{4}$

$ty = \frac{te^{4t}}{4} - \int \frac{e^{4t}}{4} dt = \frac{te^{4t}}{4} - \frac{e^{4t}}{16} + C$

Thus $y = \frac{e^{4t}}{4} - \frac{e^{4t}}{16t} + \frac{C}{t}$

Answer: $y = \frac{e^{4t}}{4} - \frac{e^{4t}}{16t} + \frac{C}{t}$

[20] 6.) Choose one of the following 2 problems. If you do not choose your best problem, I will substitute the other problem, but with a 2 point penalty (if it improves your grade). Circle the letter corresponding to your chosen problem: A B

6A.) Show that $\phi(t) = \sum_{k=2}^{\infty} \frac{3(-1)^k\ t^k}{k!}$ converges for all t and show that $\phi(t) = \sum_{k=2}^{\infty} \frac{3(-1)^k\ t^k}{k!}$ is a solution to $y' = 3t - y$.

Proof: $$\lim_{k\to\infty} \frac{\frac{3(-1)^{k+1}\ t^{k+1}}{(k+1)!}}{\frac{3(-1)^k\ t^k}{k!}} = \lim_{k\to\infty} \frac{t}{k+1} = 0 < 1 \text{ for all } t.$$

Thus $\phi(t) = \sum_{k=2}^{\infty} \frac{3(-1)^k\ t^k}{k!}$ converges for all t by the ratio test.

To show that $y = \phi(t)$ is a solution to $y' = 3t - y$, we plug it in.

$$\phi' = \sum_{k=2}^{\infty} \frac{3(-1)^k\ (k)t^{k-1}}{k!} = \sum_{k=2}^{\infty} \frac{3(-1)^k\ t^{k-1}}{(k-1)!}$$

$$3t - \phi = 3t - \sum_{k=2}^{\infty} \frac{3(-1)^k\ t^k}{k!} = 3t - \sum_{k=3}^{\infty} \frac{3(-1)^{k-1}\ t^{k-1}}{(k-1)!}$$

$$\phi' = \sum_{k=2}^{\infty} \frac{3(-1)^k\ (k)t^{k-1}}{k!} = \sum_{k=2}^{\infty} \frac{3(-1)^k\ t^{k-1}}{(k-1)!} = \sum_{k=2}^{2} \frac{3(-1)^k\ t^{k-1}}{(k-1)!} + \sum_{k=3}^{\infty} \frac{3(-1)^k\ t^{k-1}}{(k-1)!}$$

$$= 3t - \sum_{k=3}^{\infty} \frac{3(-1)^{k-1}\ t^{k-1}}{(k-1)!} = 3t - \phi$$

Thus $\phi(t) = \sum_{k=2}^{\infty} \frac{3(-1)^k\ t^k}{k!}$ is a solution to $y' = 3t - y$.

6B.) Show by induction that for Picard's iteration method, $\phi_n(t) = \sum_{k=1}^{n} \frac{3(-1)^{k+1}\ t^{k+1}}{(k+1)!}$ approximates the solution to the initial value problem, $y' = 3t - y$, $y(0) = 0$ where $\phi_1(t) = \frac{3t^2}{2}$. You may use the proof outline below or write it from scratch.

Proof by induction on n.

For $n = 1$, $\sum_{k=1}^{1} \frac{3(-1)^{k+1}\ t^{k+1}}{(k+1)!} = \frac{3(-1)^{1+1}\ t^{1+1}}{(1+1)!} = \frac{3\ t^2}{2!} = \phi_1$

Suppose for $n = j-1$, $\phi_{j-1}(t) = \sum_{k=1}^{j-1} \frac{3(-1)^{k+1}\ t^{k+1}}{(k+1)!}$

Then by Picard's iteration method, $\phi_j = \int_0^t f(s, \phi_{j-1}(s))ds$ where $f(t,y) = 3t - y$.

Thus $\phi_j = \int_0^t (3s - \phi_{j-1}(s))ds = \int_0^t (3s - \sum_{k=1}^{j-1} \frac{3(-1)^{k+1}\ s^{k+1}}{(k+1)!})ds$

$$= \int_0^t 3s\ ds - \sum_{k=1}^{j-1} \int_0^t \frac{3(-1)^{k+1}\ s^{k+1}}{(k+1)!} ds$$

$$= \tfrac{3}{2}s^2|_0^t - \sum_{k=1}^{j-1} \frac{3(-1)^{k+1}\ s^{k+2}}{(k+2)!}|_0^t$$

$$= \tfrac{3}{2}t^2 - 0 - (\sum_{k=1}^{j-1} \frac{3(-1)^{k+1}\ t^{k+2}}{(k+2)!} - 0)$$

$$= \tfrac{3}{2}t^2 + \sum_{k=1}^{j-1} \frac{3(-1)^{k+2}\ t^{k+2}}{(k+2)!}$$

$$= \sum_{k=1}^{1} \frac{3(-1)^{k+1}\ t^{k+1}}{(k+1)!} + \sum_{k=2}^{j} \frac{3(-1)^{k+1}\ t^{k+1}}{(k+1)!}$$

$$= \sum_{k=1}^{j} \frac{3(-1)^{k+1}\ t^{k+1}}{(k+1)!}$$