Problem Session Thursday 3 pm

Exam 2 review:

To solve a single differential equation, for exam 2, use Ch 5 methods:

A.) If you have an Euler equation, $x^2y'' + \alpha xy' + \beta y = 0$ where α, β are constants, use simple 5.4 method (guess $y = |x|^r$, breaks into standard 3 cases, see 5.4 handouts).

B.) Suppose you are interested in the solution near $x = x_0$, then we can find

(1.) exact solution solution by solving for the series solution (ex: see 5.2 handout)

(2.) An approximate solution by determining the first few terms in the series solution (ex: see 5.5 part 2 handout)

Determine if x_0 is an ordinary point, regular singular value, or irregular singular value.

If x_0 is an ordinary point, solution near x_0 is $\Sigma_{n=0}^{\infty} a_n(x - x_0)^n$.

If x_0 is a regular singular point, solution near x_0 is $\Sigma_{n=0}^{\infty} a_n(x - x_0)^{n+r}$.

When (and where) do you know when solution exists?

What are the subparts of these problems?

Look at theory including existence, uniqueness, domain of solution, linearity.

To solve a system of differential equations use Ch 7 methods:

Linear: find eigenvalues, eigenvectors, breaks into standard 3 cases (plus a subcase) – see last 7.5 handout

When do you know a solution exists? uniqueness? Linearity properties?

Be able to translate an *nth* order linear differential equation into a system of n linear differential equations and write in matrix form.

Understand and be able to identify different types of critical points (equilibrium solutions = constant solutions) for both linear and non-linear systems.

* asymptotically stable, stable, unstable
* sink, center, source
* spiral, node, saddle

Be able to graph phase portrait of a linear system of DE (trajectories in x_1, x_2-plane). Also be able to graph x_i versus t for simple cases.

Completely understand Fig 9.1.9.

Look at theory including existence, uniqueness, domain of solution, linearity.

$x' = A\vec{x}$

Solve $\mathbf{X}'(t) = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}(t)$

Step 1. Find eigenvalues:

$$A - \lambda I = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda) - 12$$
$$= \lambda^2 - 6\lambda + 5 - 12 = \lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1) = 0$$

Thus $\lambda = 7, -1$

Step 2. Find eigenvectors:

$\lambda = 7$: $\quad A - 7I = \begin{bmatrix} 1-7 & 3 \\ 4 & 5-7 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$

Note $\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Note the dimension of the nullspace of $\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$ is 1.

Or in other words, solution space for

$$\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ is 1-dimensional}$$

Thus a basis for the eigenspace for $\lambda = 7$ is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an e. vector for $\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = A$

$A\vec{v} = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ✓

$\lambda = -1 \quad A - (-1)I = \begin{bmatrix} 1+1 & 3 \\ 4 & 5+1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

Note $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Thus a basis for the eigenspace for $\lambda = -1$ is $\left\{ \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$

Thus a basis for the solution space to $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} \right\}$$

Hence the general solution is

$$\boxed{\mathbf{X}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t}}$$

Note we can take any basis for the solution space to create the general solution

Alternate basis: $\left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{7t}, \begin{bmatrix} -9 \\ 6 \end{bmatrix} e^{-t} \right\}$

Alternate format of general solution:

$$\mathbf{X}(t) = c_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} -9 \\ 6 \end{bmatrix} e^{-t}$$

IVP: $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}, \quad \mathbf{X}(t_0) = \begin{bmatrix} e \\ f \end{bmatrix}$

$$\begin{bmatrix} e \\ f \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t_0} + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t_0} = \begin{bmatrix} c_1e^{7t_0} + 3c_2e^{-t_0} \\ 2c_1e^{7t_0} - 2c_2e^{-t_0} \end{bmatrix}$$

Solve using any method you like. We will use matrix form:

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Solution exists if Wronskian evaluated at t_0 is not zero.

$$W\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} \right) = \begin{vmatrix} e^{7t} & 3e^{-t} \\ 2e^{7t} & -2e^{-t} \end{vmatrix}$$

$$= -2e^{6t} - 6e^{6t} = -8e^{6t} \neq 0$$

Section 7.7

Fundamental matrix: $\Phi(t) = \begin{bmatrix} e^{7t} & 3e^{-t} \\ 2e^{7t} & -2e^{-t} \end{bmatrix}$

Back to IVP: $\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix}^{-1} \begin{bmatrix} e \\ f \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix}^{-1} \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix}^{-1} \begin{bmatrix} e \\ f \end{bmatrix}$

Extra

part b of 7.7

But I would prefer a fundamental matrix whose inverse is easier to calculate, at least when $t_0 = 0$.

Thus we will find another basis for the solution set to $\mathbf{x}' = A\mathbf{x}$ so that the corresponding fundamental matrix has the property that $\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the 2x2 identity matrix.

$\boxed{\textbf{Step 1: Solve IVP:} \quad \mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$ ← part b #7.7

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^0 = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \text{ implies } \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(-\tfrac{1}{8}) \begin{bmatrix} -2 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix} \quad \& \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Thus IVP solution where $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is

$$\mathbf{X}(t) = \frac{1}{4} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t} + \frac{1}{4} \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} = \boxed{\begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} \end{bmatrix}}$$

Solve to IVP
$\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

A soln

part 6 7 7

Step 2: Solve IVP: $\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^0 = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ -\frac{1}{8} \end{bmatrix}$

2nd sol to IVP $X' = AX$, $X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Thus IVP solution where $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is

$$\mathbf{X}(t) = \frac{3}{8} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t} - \frac{1}{8} \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} = \boxed{\begin{bmatrix} \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix}}$$

Thus another basis for the solution space to $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}$

is $\left\{ \begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} \end{bmatrix}, \begin{bmatrix} \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix} \right\}$

Its corresponding fundamental matrix is

$$\begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} & \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} & \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix}$$

General sol

$$\vec{X} = c_1 \begin{bmatrix} e^{7t}/4 + 3e^{-t}/4 \\ e^{7t}/2 - e^{-t}/2 \end{bmatrix} + c_2 \begin{bmatrix} \frac{3e^{7t}}{8} - \frac{3e^{-t}}{8} \\ \frac{3e^{7t}}{4} + \frac{e^{-t}}{4} \end{bmatrix}$$

Thus to solve IVP where $\mathbf{X}(t_0) = \begin{bmatrix} e \\ f \end{bmatrix}$, we solve

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} \frac{1}{4}e^{7t_0} + \frac{3}{4}e^{-t_0} & \frac{3}{8}e^{7t_0} - \frac{3}{8}e^{-t_0} \\ \frac{1}{2}e^{7t_0} - \frac{1}{2}e^{-t_0} & \frac{3}{4}e^{7t_0} + \frac{1}{4}e^{-t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

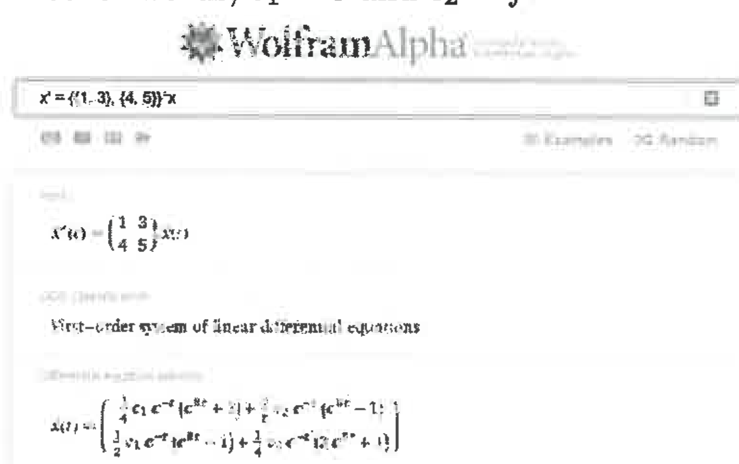
When $t_0 = 0$. I.e., we have an IVP where $\mathbf{X}(0) = \begin{bmatrix} e \\ f \end{bmatrix}$

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} \frac{1}{4}e^{0} + \frac{3}{4}e^{0} & \frac{3}{8}e^{0} - \frac{3}{8}e^{0} \\ \frac{1}{2}e^{0} - \frac{1}{2}e^{0} & \frac{3}{4}e^{0} + \frac{1}{4}e^{0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

In other words, $c_1 = e$ and $c_2 = f$.



Wolfram Alpha's soln

Ch 7 and 9

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by

$$x_1'(t) = ax_1 + bx_2,$$
$$x_2'(t) = cx_1 + dx_2$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

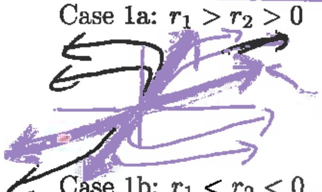
$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc = 0.$$

Thus $r = \frac{(a+d)\pm\sqrt{(a+d)^2-4(ad-bc)}}{2}$

Case 1: $(a+d)^2 - 4(ad-bc) > 0$ two real e. values

Hence the general solutions is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{r_2 t}$

Case 1a: $r_1 > r_2 > 0$



$x_2 = \frac{v_2}{v_1} x_1$

nodal source

Case 1b: $r_1 < r_2 < 0$

nodal sink

Case 1c: $r_2 < 0 < r_1$

saddle

~~**Case 2:** $(a+d)^2 - 4(ad-bc) = 0$~~ 1 repeated

~~Case 2i: Two independent eigenvectors:~~

~~The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{rt}$~~

~~Case 2ii: One independent eigenvectors:~~

~~The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} t + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right] e^{rt}$~~

~~Case 2a: $r > 0$~~

~~Case 2b: $r < 0$~~

Case 3: $(a+d)^2 - 4(ad-bc) < 0$. I.e., $r = \lambda \pm i\mu$ 2 complex e. value

Suppose the eigenvector corresponding to this eigenvalue is

$$\begin{pmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Then general solution is $\vec{X} = c_1(\vec{V}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 cos(\mu t) - w_1 sin(\mu t) \\ v_2 cos(\mu t) - w_2 sin(\mu t) \end{pmatrix} e^{\lambda t} + c_2 \begin{pmatrix} v_1 sin(\mu t) + w_1 cos(\mu t) \\ v_2 sin(\mu t) + w_2 cos(\mu t) \end{pmatrix} e^{\lambda t}$$

Case 3a: $\lambda > 0$

spiral sou

Case 3a: $\lambda < 0$

spiral sink

Case 3a: $\lambda = 0$

center

Solve: $\vec{x}' = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}$ has e.vectors $c_1 \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ w/ e.value $= -1$

and e.vectors $c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ w/ e.value $= 5$

Thus general solution is

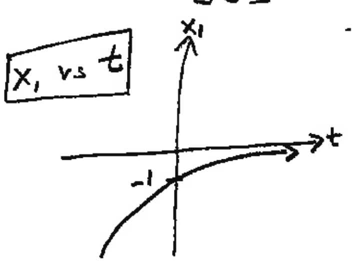
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}$$

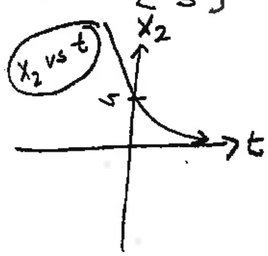
I.V.P.: Suppose $\vec{x}(0) = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

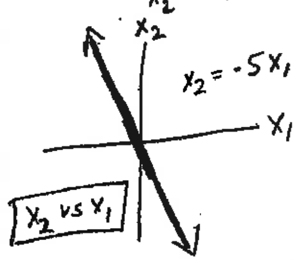
$$\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \vec{x}(0) = c_1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} e^{0} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{0}$$

$$\left.\begin{aligned} -1 &= -c_1 + c_2 \\ 5 &= 5c_1 + c_2 \end{aligned}\right\} \Rightarrow c_1 = 1, \quad c_2 = 0$$

If $\vec{x}(0) = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} e^{-t} \Rightarrow \begin{aligned} x_1 &= -e^{-t} \\ x_2 &= 5e^{-t} \end{aligned}$







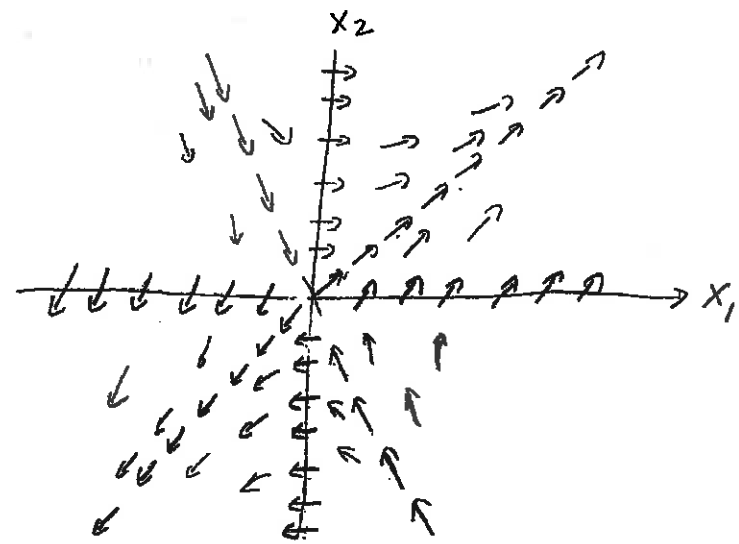
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4x_1 + x_2 \\ 5x_1 \end{bmatrix}$$

$\frac{dx_1}{dt} = 4x_1 + x_2$

$\frac{dx_2}{dt} = 5x_1$

$$\frac{dx_2}{dx_1} = \frac{dx_2}{dt} \cdot \frac{dt}{dx_1} = \frac{dx_2/dt}{dx_1/dt} = \frac{x_2'}{x_1'}$$

$$= \boxed{\frac{5x_1}{4x_1 + x_2}}$$



If $x_2 = -5x_1 \Rightarrow \frac{x_2'}{x_1'} = \frac{5x_1}{4x_1 - 5x_1} = \frac{5x_1}{-x_1} = -5$

(a) Node if $q > 0$ and $\Delta \geq 0$; (b) Saddle point if $q < 0$;

(c) Spiral point if $p \neq 0$ and $\Delta < 0$; (d) Center if $p = 0$ and $q > 0$.

Hint: These conclusions can be reached by studying the eigenvalues r_1 and r_2. It may also be helpful to establish, and then to use, the relations $r_1 r_2 = q$ and $r_1 + r_2 = p$.

21. Continuing Problem 20, show that the critical point $(0, 0)$ is

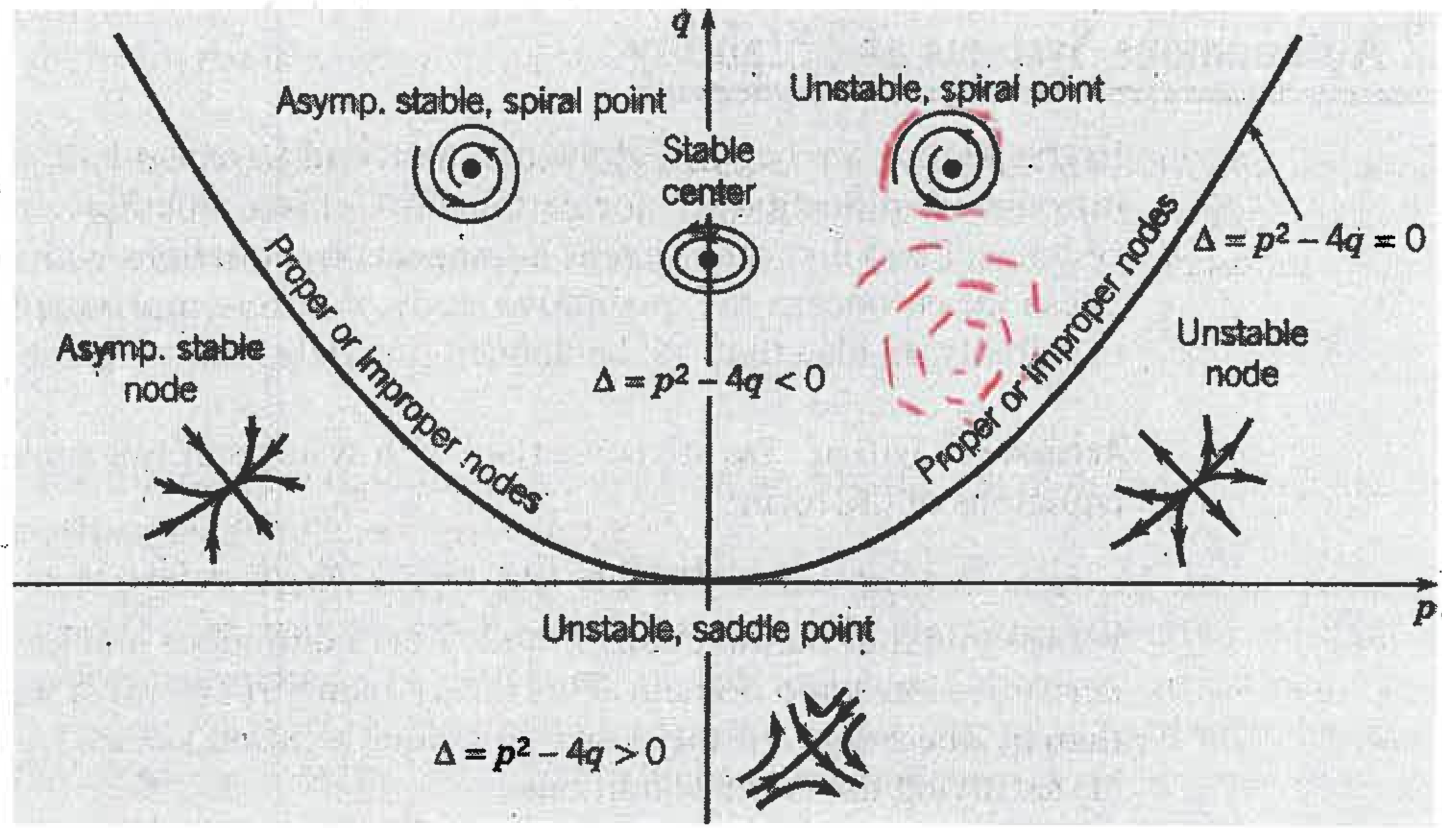
(a) Asymptotically stable if $q > 0$ and $p < 0$;

(b) Stable if $q > 0$ and $p = 0$;

(c) Unstable if $q < 0$ or $p > 0$.

The results of Problems 20 and 21 are summarized visually in Figure 9.1.9.

$q = ad - bc$



$p = a + d$

FIGURE 9.1.9 Stability diagram.

9.3 Locally linear systems.

Just like in Calc 1, we are interested in critical points. In 22M:100, these are equilibrium solutions, $(x(t), y(t)) = (x_0, y_0)$, i.e constant solutions in (x, y, t) space which thus project to a point in the (x, y) phase plane.

Just like in Calc 1, we will use linear approximations to determine what solutions look like near the critical point in the (x, y) phase plane.

The visualization when x and y are functions of t when given an autonomous system of differential equations $\frac{dx}{dt} = F(x, y)$, $\frac{dy}{dt} = G(x, y)$ is much more interesting than the calc 1 case. It is also different than the case when z is a function of t and s.

Goal: Find linear approximation for autonomous system of 1^{st} order D. E.:

$$x' = F(x, y),\ y' = G(x, y)$$

And use this linear approximation to determine what a trajectory $(x(t), y(t))$, $t \in (a, b)$ looks like near a critical point.

By Taylor's theorem, if F and G have continuous partial derivatives up to order two, then

$$F(x, y) = F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) + \eta_1(x, y)$$

$$G(x, y) = G(x_0, y_0) + G_x(x_0, y_0)(x - x_0) + G_y(x_0, y_0)(y - y_0) + \eta_2(x, y)$$

where $lim_{(x,y)\to(x_0,y_0)} \frac{\eta_i(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$

Since (x_0, y_0) is a critical point, $F(x_0, y_0) = 0$ and $G(x_0, y_0) = 0$. Thus

$$\frac{d}{dt}\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix} = \begin{bmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \begin{bmatrix} \eta_1(x, y) \\ \eta_2(x, y) \end{bmatrix}$$

Note $\begin{bmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{bmatrix}$ is a matrix with constant entries.

Let $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$

Then a **translated** linear approximation of the above matrix D. E. equation is

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Note by substituting $u_1 = x - x_0$, $u_2 = y - y_0$, we have **translated** our critical point (x_0, y_0) in the x, y-phase plane to the origin $(0, 0)$ in the u_1, u_2-phase plane.

Defn: $J(x, y) = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}$ is the *Jacobian matrix* of F and G.

Defn: $det(J(x, y))$ is the *Jacobian*.

We will restrict to the case $det(J(x_0, y_0)) \neq 0$ for the following reasons:

Just like in Calc 1, we will restrict to only looking at isolated critical points. That is we can find an ϵ−ball around (x_0, y_0) such that this ϵ−ball does not contain any other critical points. That is,
(x_0, y_0) is the unique critical point in the region defined by $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \epsilon$

In other words, if $\mathbf{x}' = A\mathbf{x}$ is a linear approximation to the autonomous system of first order D. E.: $x' = F(x, y)$, $y' = G(x, y)$ near (x_0, y_0), then we will restrict to the case when $det(A) \neq 0$.

The stability of the critical point $(x, y) = (x_0, y_0)$ of the system $x' = F(x, y)$, $y' = G(x, y)$ will depend on both

(1.) The stability of the critical point $(x, y) = (0, 0)$ of its translated linear approximation $\mathbf{x}' = A\mathbf{x}$.

(2.) How stable is the stability of the critical point $(x, y) = (0, 0)$ of its translated linear approximation $\mathbf{x}' = A\mathbf{x}$.

Example: $\quad x' = x - xy,\ \ y' = -y + xy \qquad$ Eqn (*)

critcal points: $x' = x - xy = x(1 - y) = 0$ implies $x = 0$ or $y = 1$,
$\quad y' = -y + xy = y(-1 + x) = 0$ implies $y = 0$ or $x = 1$.

Break into cases using 1rst equation (or any equation of your choice)

Case 1: If $x = 0$, then $y = 0$ by second case. Thus (0, 0) is a critical point.
Case 2: If $y = 1$, then $x = 1$ by second case. Thus (1, 1) is a critical point.

$x' = x - xy, \;\; y' = -y + xy$ Eqn (*)

Jacobian matrix: $\begin{bmatrix} 1-y & -x \\ y & -1+x \end{bmatrix}$

Case 1: $(x_0, y_0) = (0, 0)$.

The linear approximation to non-linear differential equation (*) is

$$\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$$

Determine stability of critical point (0, 0) of linear approximation:

Short method: *eigenvalues* $= 1, -1$, thus (0, 0) is an unstable saddle point of the linear approximation. If we slightly perturb (0, 0), we still have an unstable saddle point.

Thus $(x_0, y_0) = (0, 0)$ is an unstable saddle point of the nonlinear system of D.E., $x' = x - xy, \;\; y' = -y + xy$

Longer method: $det(A - rI) = \begin{vmatrix} 1-r & 0 \\ 0 & -1-r \end{vmatrix} = (1-r)(-1-r) = r^2 - 1 = 0$

Thus $r = \frac{0 \pm \sqrt{0^2 - 4(-1)}}{2} = \frac{p \pm \sqrt{p^2 - 4(q)}}{2}$ where $r^2 - pr + q = 0$. I.e., $p = 0$, $q = -1$. See figure 9.1.9 in your text.

Thus (0, 0) is an unstable saddle point of the linear approximation. If we slightly perturb the linear approximation differential equation, then we will still have one positive and one negative eigenvalue (since values close to the eigenvalue 1 will still be positive and values close to the eigenvalue -1 will still be negative. Thus we still have an unstable saddle point. Similarly, in figure 9.1.9 when $p = 0$, $q = -1$, we get an unstable saddle point for small perturbations of p and q.

Thus $(x_0, y_0) = (0, 0)$ is an unstable saddle point of nonlinear system of D.E.,

$$x' = x - xy, \;\; y' = -y + xy$$

Case 2: $(x_0, y_0) = (1, 1)$.

The linear translated approximation to non-linear differential equation (*) is

$$\mathbf{x}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

$det(A - rI) = \begin{vmatrix} -r & -1 \\ 1 & -r \end{vmatrix} = r^2 + 1 = 0$. Thus $r = \frac{0 \pm \sqrt{0^2 - 4(1)}}{2} = \pm i$.

Thus (0, 0) is a stable center point of the linear translated approximation.

If we take complex numbers, $a \pm bi$ close to the eigenvalues $r = \pm i$, then b will be close to 1 and thus positive, but a will be close to 0 and thus could be positive or negative or 0.

Hence for the nonlinear equation (*), the critical point (1, 1) is one of the following
1.) stable center point
2.) unstable spiral point
3.) asymptotically stable spiral point.

Alternatively, see figure 9.1.9 in your text where $p = 0$ and $q = 1$ since $r^2 + 1 = 0$.

± Exact Equation
2.6

Solving first order differential equation:

Method 1 (sect. 2.2): Separate variables.

Method 2 (sect. 2.1): If linear $[y'(t)+p(t)y(t) = g(t)]$, multiply equation by an integrating factor $u(t) = e^{\int p(t)dt}$.

Check

$$\begin{gathered} y' + py = g \\ y'u + upy = ug \\ \boxed{(uy)' = ug} \\ \int (uy)' = \int ug \\ uy = \int ug \\ \text{etc...} \end{gathered}$$

Method 3 (sect. 2.4): Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when $n > 1$ by changing it to a linear equation by substituting $v = y^{1-n}$

If $v = \frac{dx}{dt}$, can use the following to simplify (especially if there are 3 variables).

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

integration techniques: u-substitution, integration by parts, partial fractions.

---

direction field = slope field = graph of $\frac{dv}{dt}$ in t, v-plane.

*** can use slope field to determine behavior of v including as $t \to \infty$.
Equilibrium Solution = constant solution

stable, unstable, semi-stable.

Solving second order differential equation:

p. 135: $y'' = f(t, y')$, $y'' = f(y, y')$,

Transform to first order: Let $v = y'$.

If needed, note $v' = \frac{dv}{dt} = \frac{dv}{dt}\frac{dy}{dy} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v$.

Note this trick sometimes helpful for first order equations.

Ch 3: linear $ay'' + by' + cy = 0$,

Need to have two independent solutions.

If ϕ_1, ϕ_2 are solutions to a LINEAR HOMOGENEOUS differential equation, $c_1\phi_1 + c_2\phi_2$ is also a solution

Existence and Uniqueness

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a,b) \to R$ and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a,b) \to R$ that satisfies the initial value problem

$$y' + p(t)y = g(t),$$
$$y(t_0) = y_0$$

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a,b) \to R$, $q : (a,b) \to R$, and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a,b) \to R$ that satisfies the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),$$
$$y(t_0) = y_0, \quad y'(t_0) = y'_0$$

Definition: The Wronskian of two differential functions, f and g is

$$W(f,g) = fg' - f'g = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

d3 not d2

Thm 3.2.4: Given (1) the hypothesis of thm 3.2.1
(2) ϕ_1 and ϕ_2 are 2 sol'ns to $y'' + p(t)y' + q(t)y = 0$ (*)
(3) $W(\phi_1, \phi_2)(t_0) \neq 0$, for some $t_0 \in (a,b)$,
then if f is a solution to (*), then $f = c_1\phi_1 + c_2\phi_2$ for some c_1 and c_2.

coefficient matrix for IVP

Thm 2.4.2: Suppose $z = f(t,y)$ and $z = \frac{\partial f}{\partial y}(t,y)$ are continuous on $(a,b) \times (c,d)$ and the point $(t_0, y_0) \in (a,b) \times (c,d)$, then there exists an interval $(t_0 - h, t_0 + h) \subset (a,b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t,y), \quad y(t_0) = y_0.$$

Note the initial value problem

$$y' = y^{\frac{1}{3}}, \; y(0) = 0$$

nonlinear eqns

has an infinite number of different solutions.

$$y^{-\frac{1}{3}} dy = dt$$
$$\tfrac{3}{2} y^{\frac{2}{3}} = t + C$$
$$y = \pm(\tfrac{2}{3}t + C)^{\frac{3}{2}}$$
$$y(0) = 0 \text{ implies } C = 0$$

can be weird

Thus $y = \pm(\frac{2}{3}t)^{\frac{3}{2}}$ are solutions.

$y = 0$ is also a solution, etc.

Compare to Thm 2.4.2:
$f(t,y) = y^{\frac{1}{3}}$ is continuous near (0, 0)
But $\frac{\partial f}{\partial y}(t,y) = \frac{1}{3}y^{\frac{-2}{3}}$ is not continuous near (0, 0) since it isn't defined at (0, 0).

$y = c_1 f + c_2 g$
$y' = c_1 f' + c_2 g'$
$\rightarrow \begin{bmatrix} f & g \\ f' & g' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$

Section 2.4 example: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$F(y,t) = \frac{1}{(1-t)(2-y)}$ is continuous for all $t \neq 1$, $y \neq 2$

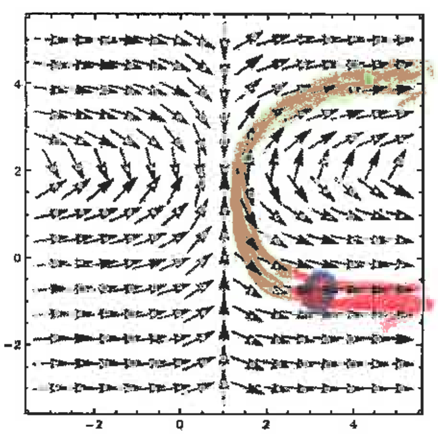
$$\frac{\partial F}{\partial y} = \frac{\partial\left(\frac{1}{(1-t)(2-y)}\right)}{\partial y} = \frac{1}{(1-t)}\frac{\partial(2-y)^{-1}}{\partial y} = \frac{1}{(1-t)(2-y)^2}$$

$\frac{\partial F}{\partial y}$ is continuous for all $t \neq 1$, $y \neq 2$

Thus the IVP $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = y_0$ has a unique solution if $t_0 \neq 1$, $y_0 \neq 2$.

Note that if $y_0 = 2$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = 2$ has two solutions if $t_0 \neq 2$

Note that if $t_0 = 1$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(1) = y_0$ has no solutions.



$(1, 1/((1-x)(2-y)))/sqrt1 + 1/((1-x)(2-y))^2)$

Solve via separation of variables:

$\int(2-y)dy = \int \frac{dt}{1-t}$

$2y - \frac{y^2}{2} = -ln|1-t| + C$

$y^2 - 4y - 2ln|1-t| + C = 0$

$$y = \frac{4 \pm \sqrt{16 + 4(2ln|1-t| + C)}}{2} = 2 \pm \sqrt{4 + 2ln|1-t| + C}$$

$$y = 2 \pm \sqrt{2ln|1-t| + C}$$

Find domain: $2ln|1-t| + C \geq 0$

$2ln|1-t| \geq -C$

$ln|1-t| \geq -\frac{C}{2}$ Note: we want to find domain for this C and thus this C can't swallow constants).

$|1-t| \geq e^{-\frac{C}{2}}$ since e^x is an increasing function.

$1 - t \leq -e^{-\frac{C}{2}}$ or $1 - t \geq e^{-\frac{C}{2}}$

$-t \leq -e^{-\frac{C}{2}} - 1$ or $-t \geq e^{-\frac{C}{2}} - 1$

Domain: $\begin{cases} t \geq e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 0 \\ t \leq -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 0. \end{cases}$

Note: Domain is much easier to determine when the ODE is linear.

Find C given $y(t_0) = y_0$: $y_0 = 2 \pm \sqrt{2ln|1-t_0|+C}$

$\pm(y_0 - 2) = \sqrt{2ln|1-t_0|+C}$

$(y_0-2)^2 - 2ln|1-t_0| = C$

$y = 2 \pm \sqrt{2ln|1-t|+C}$

$y = 2 \pm \sqrt{2ln|1-t| + (y_0-2)^2 - 2ln|1-t_0|}$

$y = 2 \pm \sqrt{(y_0-2)^2 + ln\frac{(1-t)^2}{(1-t_0)^2}}$

Domain: $\begin{cases} t \geq e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 0 \\ t \leq -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 0. \end{cases}$

$e^{-\frac{C}{2}} = e^{-\frac{(y_0-2)^2 - 2ln|1-t_0|}{2}} = |1-t_0|e^{-\frac{(y_0-2)^2}{2}}$

Domain: $\begin{cases} t \geq 1 + |1-t_0|e^{-\frac{(y_0-2)^2}{2}} & \text{if } t_0 > 0 \\ t \leq 1 - |1-t_0|e^{-\frac{(y_0-2)^2}{2}} & \text{if } t_0 < 0. \end{cases}$

Section 2.5:

Exponential Growth/Decay
Example: population growth/radioactive decay)

$y' = ry$, $y(0) = y_0$ implies $y = y_0 e^{rt}$

$r > 0$ $r < 0$

Logistic growth: $y' = h(y)y$

Example: $y' = r(1 - \frac{y}{K})y$

y vs $f(y)$ slope field:

Equilibrium solutions:

Asymptotically stable:

Asymptotically unstable:

Asymptotically semi-stable:

As $t \to \infty$, if $y > 0$, $y \to$

Solution: $y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$

2.8 Approx a soln

Given: $y' = f(t, y)$, $y(0) = 0$ Eqn (*)

f, $\partial f/\partial y$ continuous $\forall (t, y) \in (-a, a) \times (-b, b)$. Then

$y = \phi(t)$ is a solution to (*) iff

$\boxed{\phi'(t) = f(t, \phi(t))}$, $\phi(0) = 0$ iff

$\boxed{\int_0^t \phi'(s)ds = \int_0^t f(s, \phi(s))ds}$, $\phi(0) = 0$ iff

$\boxed{\phi(t)} = \phi(t) - \phi(0) = \int_0^t f(s, \phi(s))ds$

Thus $y = \phi(t)$ is a solution to (*) iff $\phi(t) = \int_0^t f(s, \phi(s))ds$

Construct ϕ using method of successive approximation
– also called Picard's iteration method.

Let $\phi_0(t) = 0$ (or the function of your choice)

Let $\phi_1(t) = \int_0^t f(s, \phi_0(s))ds$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds$ approx ϕ soln w/ ϕ_n

$\vdots$

$\boxed{\text{Let } \phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds}$

Let $\phi(t) = \lim_{n\to\infty} \phi_n(t)$

Some questions:

1.) Does $\phi_n(t)$ exist for all n?

2.) Does sequence ϕ_n converge?

3.) Is $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ a solution to (*).

4.) Is the solution unique.

Example: $y' = t + 2y$. That is $f(t, y) = t + 2y$

Let $\phi_0(t) = 0$

Let $\phi_1(t) = \int_0^t f(s, 0)ds = \int_0^t (s + 2(0))ds$

$$= \int_0^t sds = \frac{s^2}{2}\Big|_0^t = \frac{t^2}{2}$$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds = \int_0^t f(s, \frac{s^2}{2})ds$

$$= \int_0^t (s + 2(\frac{s^2}{2}))ds = \frac{t^2}{2} + \frac{t^3}{3}$$

Let $\phi_3(t) = \int_0^t f(s, \phi_2(s))ds = \int_0^t f(s, \frac{s^2}{2} + \frac{s^3}{3})ds$

$$= \int_0^t (s + 2\frac{s^2}{2} + \frac{s^3}{3}))ds = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$$

$\vdots$

See class notes.