Solving first order differential equation:

Method 1 (sect. 2.2): Separate variables.

Method 2 (sect. 2.1): If linear $[y'(t)+p(t)y(t) = g(t)]$, multiply equation by an integrating factor $u(t) = e^{\int p(t)dt}$.

$$\begin{aligned} y' + py &= g \\ y'u + upy &= ug \\ (uy)' &= ug \\ \int (uy)' &= \int ug \\ uy &= \int ug \end{aligned}$$

etc...

Method 3 (sect. 2.4): Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when $n > 1$ by changing it to a linear equation by substituting $v = y^{1-n}$

If $v = \frac{dx}{dt}$, can use the following to simplify (especially if there are 3 variables).

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

2.5 Autonomous system

integration techniques: u-substitution, integration by parts, partial fractions.

---

direction field = slope field = graph of $\frac{dv}{dt}$ in t, v-plane.

*** can use slope field to determine behavior of v including as $t \to \infty$.
Equilibrium Solution = constant solution

stable, unstable, semi-stable.

Solving second order differential equation:

p. 135: $y'' = f(t, y')$, $y'' = f(y, y')$,

Transform to first order: Let $v = y'$.

If needed, note $v' = \frac{dv}{dt} = \frac{dv}{dt}\frac{dy}{dy} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v$.

Note this trick sometimes helpful for first order equations.

Ch 3: linear $ay'' + by' + cy = 0$,

Need to have two independent solutions.

If ϕ_1, ϕ_2 are solutions to a LINEAR HOMOGENEOUS differential equation, $c_1\phi_1 + c_2\phi_2$ is also a solution

Existence and Uniqueness

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a,b) \to R$ and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a,b) \to R$ that satisfies the initial value problem

$$y' + p(t)y = g(t),$$
$$y(t_0) = y_0$$

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a,b) \to R$, $q : (a,b) \to R$, and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a,b) \to R$ that satisfies the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),$$
$$y(t_0) = y_0, \quad y'(t_0) = y'_0$$

ch 2

Definition: The Wronskian of two differential functions, f and g is

$$W(f,g) = fg' - f'g = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

Thm 3.2.4: Given (1) the hypothesis of thm 3.2.1
(2) ϕ_1 and ϕ_2 are 2 sol'ns to $y''+p(t)y'+q(t)y = 0$ (*)
(3) $W(\phi_1, \phi_2)(t_0) \neq 0$, for some $t_0 \in (a,b)$,
then if f is a solution to (*), then $f = c_1\phi_1 + c_2\phi_2$ for some c_1 and c_2.

$y = c_1\phi_1 + c_2\phi_2$
$y' = c_1\phi_1' + c_2\phi_2'$

$\to \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} =$ matrix

correct for ch 3

In general Wronskian = det of coeff matrix of IVP

Thm 2.4.2: Suppose $z = f(t,y)$ and $z = \frac{\partial f}{\partial y}(t,y)$ are continuous on $(a,b) \times (c,d)$ and the point $(t_0, y_0) \in (a,b) \times (c,d)$, then there exists an interval $(t_0 - h, t_0 + h) \subset (a,b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t,y), \quad y(t_0) = y_0.$$

Note the initial value problem

$$y' = y^{\frac{1}{3}}, \; y(0) = 0$$

← nonlinear is weird

has an infinite number of different solutions.

$$y^{-\frac{1}{3}}dy = dt$$
$$\tfrac{3}{2}y^{\frac{2}{3}} = t + C$$
$$y = \pm(\tfrac{2}{3}t + C)^{\frac{3}{2}}$$
$$y(0) = 0 \text{ implies } C = 0$$

Thus $y = \pm(\frac{2}{3}t)^{\frac{3}{2}}$ are solutions.

$y = 0$ is also a solution, etc.

Compare to Thm 2.4.2:
$f(t,y) = y^{\frac{1}{3}}$ is continuous near (0, 0)
But $\frac{\partial f}{\partial y}(t,y) = \frac{1}{3}y^{\frac{-2}{3}}$ is not continuous near (0, 0) since it isn't defined at (0, 0).

Section 2.4 example: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$F(y,t) = \frac{1}{(1-t)(2-y)}$ is continuous for all $t \neq 1$, $y \neq 2$

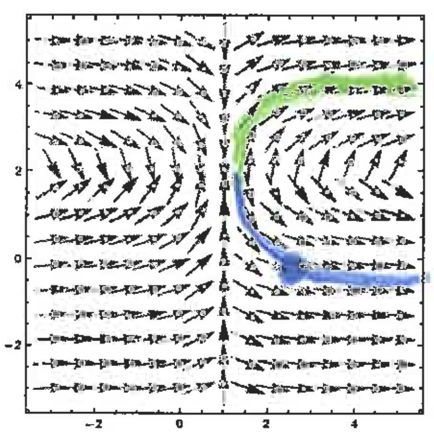
$$\frac{\partial F}{\partial y} = \frac{\partial\left(\frac{1}{(1-t)(2-y)}\right)}{\partial y} = \frac{1}{(1-t)}\frac{\partial(2-y)^{-1}}{\partial y} = \frac{1}{(1-t)(2-y)^2}$$

$\frac{\partial F}{\partial y}$ is continuous for all $t \neq 1$, $y \neq 2$

Thus the IVP $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = y_0$ has a unique solution if $t_0 \neq 1$, $y_0 \neq 2$.

Note that if $y_0 = 2$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = 2$ has two solutions if $t_0 \neq 2$

Note that if $t_0 = 1$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(1) = y_0$ has no solutions.



$(1, 1/((1-x)(2-y)))/sqrt1 + 1/((1-x)(2-y))^2)$

Solve via separation of variables:

$\int(2-y)dy = \int \frac{dt}{1-t}$

$2y - \frac{y^2}{2} = -ln|1-t| + C$

$y^2 - 4y - 2ln|1-t| + C = 0$

$y = \frac{4 \pm \sqrt{16+4(2ln|1-t|+C)}}{2} = 2 \pm \sqrt{4 + 2ln|1-t| + C}$

$$y = 2 \pm \sqrt{2ln|1-t| + C}$$

Find domain: $2ln|1-t| + C \geq 0$

$2ln|1-t| \geq -C$

$ln|1-t| \geq -\frac{C}{2}$ Note: we want to find domain for this C and thus this C can't swallow constants).

$|1-t| \geq e^{-\frac{C}{2}}$ since e^x is an increasing function.

$1 - t \leq -e^{-\frac{C}{2}}$ or $1 - t \geq e^{-\frac{C}{2}}$

$-t \leq -e^{-\frac{C}{2}} - 1$ or $-t \geq e^{-\frac{C}{2}} - 1$

Domain: $\begin{cases} t \geq e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 0 \\ t \leq -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 0. \end{cases}$

Note: Domain is much easier to determine when the ODE is linear.

Find C given $y(t_0) = y_0$**:** $y_0 = 2 \pm \sqrt{2ln|1-t_0| + C}$

$\pm(y_0 - 2) = \sqrt{2ln|1-t_0| + C}$

$(y_0 - 2)^2 - 2ln|1-t_0| = C$

$y = 2 \pm \sqrt{2ln|1-t| + C}$

$y = 2 \pm \sqrt{2ln|1-t| + (y_0 - 2)^2 - 2ln|1-t_0|}$

$y = 2 \pm \sqrt{(y_0 - 2)^2 + ln\frac{(1-t)^2}{(1-t_0)^2}}$

Domain: $\begin{cases} t \geq e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 0 \\ t \leq -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 0. \end{cases}$

$e^{-\frac{C}{2}} = e^{-\frac{(y_0-2)^2 - 2ln|1-t_0|}{2}} = |1-t_0|e^{-\frac{(y_0-2)^2}{2}}$

Domain: $\begin{cases} t \geq 1 + |1-t_0|e^{-\frac{(y_0-2)^2}{2}} & \text{if } t_0 > 0 \\ t \leq 1 - |1-t_0|e^{-\frac{(y_0-2)^2}{2}} & \text{if } t_0 < 0. \end{cases}$

Section 2.5:

Exponential Growth/Decay
Example: population growth/radioactive decay)

$y' = ry$, $y(0) = y_0$ implies $y = y_0e^{rt}$

$r > 0$

$r < 0$

Logistic growth: $y' = h(y)y$

Example: $y' = r(1 - \frac{y}{K})y$

y vs $f(y)$

slope field:

Equilibrium solutions:

Asymptotically stable:

Asymptotically unstable:

Asymptotically semi-stable:

As $t \to \infty$, if $y > 0$, $y \to$

Solution: $y = \frac{y_0K}{y_0 + (K - y_0)e^{-rt}}$

Solving polynomial equations:

Ex: $r^3 + r^2 + 3r + 10 = 0$

Plug in $r = \pm1, \pm2, \pm5, \pm10$ to see if any of these are solutions:

$(\pm1)^3 + (\pm1)^2 + 3(\pm1) + 10 \neq 0$

$(\pm2)^3 + (\pm2)^2 + 3(\pm2) + 10\ ? =?0$

$(-2)^3 + (-2)^2 + 3(-2) + 10 = -8 + 4 - 6 + 10 = 0$

Thus $(r - (-2))$ is a factor of $r^3 + r^2 + 3r + 10$

Hence $r^3 + r^2 + 3r + 10 = (r+2)(r^2 + \underline{\ \ X\ \ }\, r + 5)$

$r^3 + r^2 + 3r + 10 = (r+2)(r^2 - r + 5) = 0$ X = −1

Thus $r = -2, \frac{1 \pm \sqrt{1-20}}{2}$.   Thus $r = -2, \frac{1 \pm i\sqrt{19}}{2}$.

In special cases, you can use the unit circle.

Ex: $r^4 + 1 = 0$ implies

$r = (-1)^{\frac{1}{4}} = (-1 + 0i)^{\frac{1}{4}} = (e^{i\pi})^{\frac{1}{4}} = (e^{i(\pi + 2\pi k)})^{\frac{1}{4}}$

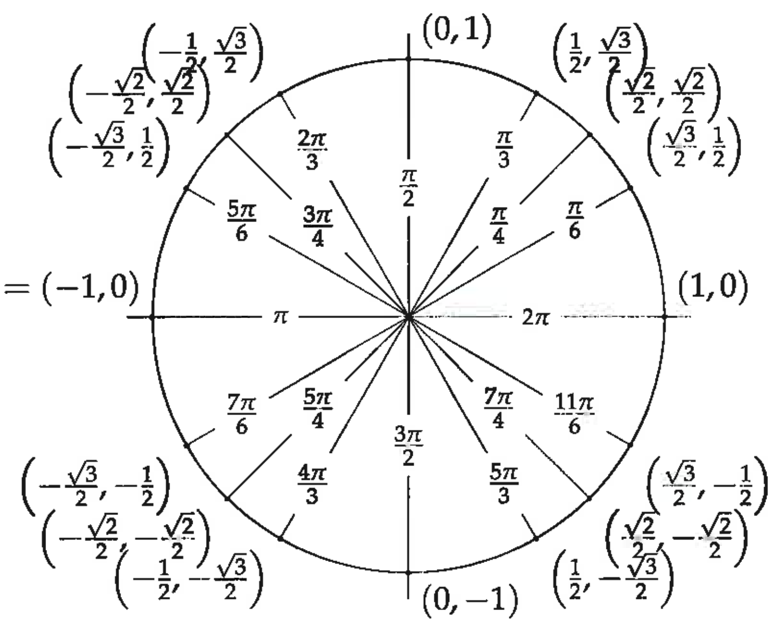
$k = 0: \quad e^{\frac{i\pi}{4}} = cos(\frac{i\pi}{4}) + isin(\frac{i\pi}{4}) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$k = 1: \quad e^{\frac{3i\pi}{4}} = cos(\frac{3i\pi}{4}) + isin(\frac{3i\pi}{4}) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$k = 2: \quad e^{\frac{5i\pi}{4}} = cos(\frac{5i\pi}{4}) + isin(\frac{5i\pi}{4}) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$k = 3: \quad e^{\frac{7i\pi}{4}} = cos(\frac{7i\pi}{4}) + isin(\frac{7i\pi}{4}) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$-1 + 0i = (-1, 0)$



Second order differential equation:

Linear equation with constant coefficients:

If the second order differential equation is

$$ay'' + by' + cy = 0,$$

then $y = e^{rt}$ is a solution

Need to have two independent solutions.

Solve the following IVPs:

1.) $y'' - 6y' + 9y = 0$ $\quad y(0) = 1,\ y'(0) = 2$

2.) $4y'' - y' + 2y = 0$ $\quad y(0) = 3,\ y'(0) = 4$

3.) $4y'' + 4y' + y = 0$ $\quad y(0) = 6,\ y'(0) = 7$

4.) $2y'' - 2y = 0$ $\quad y(0) = 5,\ y'(0) = 9$

$ay'' + by' + cy = 0, \quad y = e^{rt}$, then
$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$

If $b^2 - 4ac > 0$, general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$.

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

general solution is $y = c_1e^{dt}cos(nt) + c_2e^{dt}sin(nt)$ where $r = d \pm in$

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: te^{r_1t}

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$.

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y'_0$ to solve for c_1, c_2 to find unique solution.

Derivation of general solutions:

If $b^2 - 4ac > 0$ we guessed e^{rt} is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$e^{it} = cos(t) + isin(t)$$

Hence $e^{(d+in)t} = e^{dt} e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$= c_1 e^{dt}[cos(nt) + isin(nt)] + c_2 e^{dt}[cos(-nt) + isin(-nt)]$

$= c_1 e^{dt} cos(nt) + ic_1 e^{dt} sin(nt) + c_2 e^{dt} cos(nt) - ic_2 e^{dt} sin(nt)$

$= (c_1 + c_2) e^{dt} cos(nt) + i(c_1 - c_2) e^{dt} sin(nt)$

$= k_1 e^{dt} cos(nt) + k_2 e^{dt} sin(nt)$

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1 t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $y = v(t)e^{rt}$?

$y' = v'(t)e^{rt} + v(t)re^{rt}$

$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2 e^{rt}$

$= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2 e^{rt}$

$ay'' + by' + cy = 0$

$a(v''e^{rt} + 2v're^{rt} + vr^2 e^{rt}) + b(v'e^{rt} + vre^{rt}) + cve^{rt} = 0$

$a(v''(t) + 2v'(t)r + v(t)r^2) + b(v'(t) + v(t)r) + cv(t) = 0$

$av''(t) + 2av'(t)r + av(t)r^2 + bv'(t) + bv(t)r + cv(t) = 0$

$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$

$av''(t) + (2a(\frac{-b}{2a}) + b)v'(t) + 0 = 0$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$

$av''(t) + (-b + b)v'(t) = 0$. Thus $av''(t) = 0$.

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1 t + k_2$

Hence $v(t)e^{r_1 t} = (k_1 t + k_2)e^{r_1 t}$ is a soln

Thus $te^{r_1 t}$ is a nice second solution.

Hence general solution is $y = c_1 e^{r_1 t} + c_2 te^{r_1 t}$

Thm: Suppose $c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to

$$ay'' + by' + cy = 0,$$

If ψ is a solution to

$$ay'' + by' + cy = g(t) \ [*],$$

Then $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to [*].

Moreover if γ is also a solution to [*], then there exist constants c_1, c_2 such that

$$\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$$

Or in other words, $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to [*].

Proof:

Define $L(f) = af'' + bf' + cf$.

Recall L is a linear function.

Let $h = c_1\phi_1(t) + c_2\phi_2(t)$. Since h is a solution to the differential equation, $ay'' + by' + cy = 0$,

Since ψ is a solution to $ay'' + by' + cy = g(t)$,

We will now show that $\psi + c_1\phi_1(t) + c_2\phi_2(t) = \psi + h$ is also a solution to [*].

Since γ a solution to $ay'' + by' + cy = g(t)$,

We will first show that $\gamma - \psi$ is a solution to the differential equation $ay'' + by' + cy = 0$.

Since $\gamma - \psi$ is a solution to $ay'' + by' + cy = 0$ and

$c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to

$$ay'' + by' + cy = 0,$$

there exist constants c_1, c_2 such that

$\gamma - \psi =$ ____________________

Thus $\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$.

Thm:
Suppose f_1 is a a solution to $ay'' + by' + cy = g_1(t)$ and f_2 is a a solution to $ay'' + by' + cy = g_2(t)$, then $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$

Proof:

Since f_1 is a solution to $ay'' + by' + cy = g_1(t)$,

Since f_2 is a solution to $ay'' + by' + cy = g_2(t)$,

We will now show that $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$.

Sidenote: The proofs above work even if a, b, c are functions of t instead of constants.

Examples:

Find a suitable form for ψ for the following differential equations:

1.) $y'' - 4y' - 5y = 4e^{2t}$

2.) $y'' - 4y' - 5y = 4sin(3t)$

3.) $y'' - 4y' - 5y = t^2 - 2t + 1$

4.) $y'' - 5y = 4sin(3t)$

5.) $y'' - 4y' = t^2 - 2t + 1$

6.) $y'' - 4y' - 5y = 4(t^2 - 2t - 1)e^{2t}$

7.) $y'' - 4y' - 5y = 4sin(3t)e^{2t}$

8.) $y'' - 4y' - 5y = 4(t^2 - 2t - 1)sin(3t)e^{2t}$

9.) $y'' - 4y' - 5y = 4sin(3t) + 4sin(3t)e^{2t}$

10.) $y'' - 4y' - 5y$
$= 4sin(3t)e^{2t} + 4(t^2 - 2t - 1)e^{2t} + t^2 - 2t - 1$

11.) $y'' - 4y' - 5y = 4sin(3t) + 5cos(3t)$

12.) $y'' - 4y' - 5y = 4e^{-t}$

To solve $ay'' + by' + cy = g_1(t) + g_2(t) + ...g_n(t)$ [**]

1.) Find the general solution to $ay'' + by' + cy = 0$:
$$c_1\phi_1 + c_2\phi_2$$

2.) For each g_i, find a solution to $ay'' + by' + cy = g_i$:
$$\psi_i$$

This includes plugging guessed solution into $ay'' + by' + cy = g_i$ to find constant(s).

The general solution to [**] is

$$c_1\phi_1 + c_2\phi_2 + \psi_1 + \psi_2 + ...\psi_n$$

3.) If initial value problem:

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2).

3.6 Variation of Parameters   Solve $y'' - 2y' + y = e^t ln(t)$

1) Find homogeneous solutions: Solve $y'' - 2y' + y = 0$

Guess: $y = e^{rt}$, then $y' = re^{rt}$, $y'' = r^2e^{rt}$, and

$$r^2e^{rt} - 2re^{rt} + e^{rt} = 0 \text{ implies } r^2 - 2r + 1 = 0$$

$(r-1)^2 = 0$, and hence $r = 1$

General homogeneous solution: $y = c_1e^t + c_2te^t$

since have two linearly independent solutions: $\{e^t, te^t\}$

2.) Find a non-homogeneous solution:

Sect. 3.5 method: Educated guess

Sect. 3.6: **Guess** $y = u_1(t)e^t + u_2(t)te^t$ **and solve for** u_1 **and** u_2

$$u_1(t) = \int \frac{\begin{vmatrix} 0 & \phi_2 \\ 1 & \phi_2' \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}} g(t)dt = -\int \frac{\phi_2(t)g(t)}{W(\phi_1,\phi_2)}dt = -\int \frac{(te^t)(e^tln(t))}{e^{2t}}dt$$

$$= -\int tln(t) = -[\frac{t^2ln(t)}{2} - \int \frac{t}{2}] = -\frac{t^2ln(t)}{2} + \frac{t^2}{4}$$

$$u_2(t) = \int \frac{\begin{vmatrix} \phi_1 & 0 \\ \phi_1' & 1 \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}} g(t)dt = \int \frac{\phi_1(t)g(t)}{W(\phi_1,\phi_2)}dt = \int \frac{(e^t)(e^tln(t))}{e^{2t}}dt$$

$$= \int ln(t) = tln(t) - t$$

$$W(\phi_1,\phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}$$

$u = ln(t)$ $dv = tdt$
$du = \frac{dt}{t}$ $v = \frac{t^2}{2}$

$u = ln(t)$ $dv = dt$
$du = \frac{dt}{t}$ $v = t$

General solution : $y = c_1e^t + c_2te^t + (-\frac{t^2ln(t)}{2} + \frac{t^2}{4})e^t + (tln(t) - t)te^t$

which simplifies to $y = c_1e^t + c_2te^t + (\frac{ln(t)}{2} - \frac{3}{4})t^2e^t$

Solve $y'' + p(t)y' + q(t)y = g(t)$ where $y = c_1\phi_1(t) + c_2\phi_2(t)$ is solution to homogeneous equation $y'' + p(t)y' + q(t)y = 0$

Guess $y = u_1(t)\phi_1(t) + u_2(t)\phi_2(t)$

$y = u_1\phi_1 + u_2\phi_2$ implies $y' = u_1\phi_1' + u_1'\phi_1 + u_2\phi_2' + u_2'\phi_2$

Two unknown functions, u_1 and u_2, but only one equation ($y'' + p(t)y' + q(t)y = g(t)$). Thus might be OK to choose 2nd eq'n.

Avoid 2nd derivative in y'': **Choose** $u_1'\phi_1 + u_2'\phi_2 = 0$

$y' = u_1\phi_1' + u_2\phi_2'$ implies $y'' = u_1\phi_1'' + u_1'\phi_1' + u_2\phi_2'' + u_2'\phi_2'$

Plug into $y'' + p(t)y' + q(t)y = g(t)$:

$$u_1\phi_1'' + u_1'\phi_1' + u_2\phi_2'' + u_2'\phi_2' + p(u_1\phi_1' + u_2\phi_2') + q(u_1\phi_1 + u_2\phi_2) = g$$

$$u_1\phi_1'' + u_1'\phi_1' + u_2\phi_2'' + u_2'\phi_2' + pu_1\phi_1' + pu_2\phi_2') + qu_1\phi_1 + qu_2\phi_2 = g$$

$$u_1\phi_1'' + pu_1\phi_1' + qu_1\phi_1 + u_1'\phi_1' + u_2\phi_2'' + pu_2\phi_2' + qu_2\phi_2 + u_2'\phi_2' = g$$

$$u_1(\phi_1'' + p\phi_1' + q\phi_1) + u_1'\phi_1' + u_2(\phi_2'' + p\phi_2' + q\phi_2) + u_2'\phi_2' = g$$

ϕ_1, ϕ_2 are homogeneous solutions. Thus $\phi_i'' + p\phi_i' + q\phi_i = 0$.

Hence $u_1(0) + u_1'\phi_1' + u_2(0) + u_2'\phi_2' = g$

Thus we have 2 eqns to find 2 unknowns, the functions u_1 and u_2:

$$\begin{array}{l} u_1'\phi_1 + u_2'\phi_2 = 0 \\ u_1'\phi_1' + u_2'\phi_2' = g \end{array} \text{ implies } \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix}$$

Cramer's rule: $u_1'(t) = \frac{\begin{vmatrix} 0 & \phi_2 \\ g & \phi_2' \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}}$ and $u_2'(t) = \frac{\begin{vmatrix} \phi_1 & 0 \\ \phi_1' & g \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}}$

Sect.3.6: **Guess** $y = u_1(t)e^t + u_2(t)te^t$ **and solve for** u_1 **and** u_2

$y' = u_1'e^t + u_1e^t + u_2'te^t + u_2(e^t + te^t) = e^{2t} + te^{2t} - te^{2t} - e^{2t}$.

Two unknown functions, u_1 and u_2, but only one equation ($y'' - 2y' + y = e^t ln(t)$). Thus might be OK to choose 2nd eq'n.

Avoid 2nd derivative in y''**:** **Choose** $u_1'e^t + u_2'te^t = 0$

Hence $y' = u_1e^t + u_2(e^t + te^t)$.

and $y'' = u_1'e^t + u_1e^t + u_2'(e^t + te^t) + u_2(e^t + e^t + te^t)$.

$= u_1'e^t + u_1e^t + u_2'e^t + u_2'te^t + u_2(2e^t + te^t)$.

$= u_1e^t + u_2'e^t + u_2(2e^t + te^t)$.

Solve $y'' - 2y' + y = e^t ln(t)$

$u_1e^t + u_2'e^t + u_2(2e^t + te^t) - 2[u_1e^t + u_2(e^t + te^t)] + u_1e^t + u_2te^t = e^t ln(t)$

$u_2'e^t + 2u_2e^t + u_2te^t - 2u_2e^t - 2u_2te^t + u_2te^t = e^t ln(t)$

$u_2' = ln(t)$ or in other words, $\frac{du_2}{dt} = ln(t)$

Thus $\int du_2 = \int ln(t)dt$

$u_2 = tln(t) - t$. Note only need one solution, so don't need $+C$.

$y = u_1(t)e^t + [tln(t) - t]te^t$

$u_1'e^t + u_2'te^t = 0$. Thus $u_1' + u_2't = 0$. Hence $u_1' = -u_2't = -tln(t)$

Thus $u_1 = -\int tln(t)dt = -\frac{t^2 ln(t)}{2} + \frac{t^2}{4}$

Thus the general solution is

$$y = c_1e^t + c_2te^t + (-\frac{t^2 ln(t)}{2} + \frac{t^2}{4})e^t + (tln(t) - t)te^t$$

Section 2.7 Euler method: Using tangent lines to approximate a function.

$$y_{i+1} = y_i + \Delta y = y_i + \frac{\Delta y}{\Delta t}\Delta t \cong y_i + \frac{dy}{dt}\Delta t$$

Alternatively use equation of tangent line:

$$\text{slope} = \frac{y_{i+1}-y_i}{t_{i+1}-t_i} = f'(y_i, t_i).$$

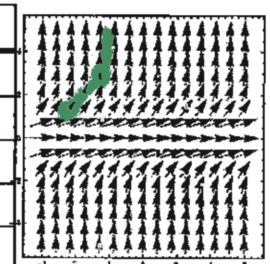
$y_{i+1} = f'(y_i, t_i)(t_{i+1} - t_i) + y_i = T(t_{i+1})$ where $y = T(t)$ is the equation of the tangent line at (y_i, t_i).

Example: $\frac{dy}{dt} = y^2$, $y(2) = 1$ implies $y = \frac{1}{3-t}$.

t	$y = 1/(3-t)$	approximation
2.000000	1.000000	1.000000
3.000000	999.000000	2.000000
4.000000	-1.000000	6.000000
5.000000	-0.500000	42.000000
6.000000	-0.333333	1806.000000

t	$y = 1/(3-t)$	approximation
2.000000	1.000000	1.000000
2.100000	1.111111	1.100000
2.200000	1.250000	1.221000
2.300000	1.428571	1.370084
2.400000	1.666667	1.557797
2.500000	2.000000	1.800470
2.600000	2.500000	2.124640
2.700000	3.333333	2.576049
2.800000	5.000000	3.239652
2.900000	10.000004	4.289186

t	$y = 1/(3-t)$	approximation
2.00	1.000000	1.000000
2.01	1.010101	1.010000
2.02	1.020408	1.020201
2.03	1.030928	1.030609
2.04	1.041667	1.041231
2.05	1.052632	1.052072
2.06	1.063830	1.063141
2.07	1.075269	1.074443
2.08	1.086957	1.085988
2.09	1.098901	1.097782
2.10	1.111111	1.109833
2.11	1.123595	1.122150
2.12	1.136364	1.134742
2.13	1.149425	1.147619
2.87	7.692308	6.721314
2.88	8.333333	7.173075
2.89	9.090908	7.687605
2.90	9.999998	8.278598
2.91	11.111107	8.963949
2.92	12.499993	9.767473
2.93	14.285716	10.721509
2.94	16.666666	11.871017
2.95	19.999996	13.280227
2.96	24.999987	15.043871
2.97	33.333298	17.307051
2.98	49.999897	20.302391
2.99	99.999496	24.424261



$y' = y^2$

2.8

Given: $y' = f(t,y)$, $y(0) = 0$ Eqn (*)

f, $\partial f/\partial y$ continuous $\forall (t,y) \in (-a,a) \times (-b,b)$. Then

$y = \phi(t)$ is a solution to (*) iff

$\phi'(t) = f(t, \phi(t))$, $\phi(0) = 0$ iff

$\int_0^t \phi'(s)ds = \int_0^t f(s, \phi(s)ds$, $\phi(0) = 0$ iff

$\phi(t) = \phi(t) - \phi(0) = \int_0^t f(s, \phi(s)ds$

Thus $y = \phi(t)$ is a solution to (*) iff $\phi(t) = \int_0^t f(s, \phi(s)ds$

Construct ϕ using method of successive approximation – also called Picard's iteration method.

Let $\phi_0(t) = 0$ (or the function of your choice)

Let $\phi_1(t) = \int_0^t f(s, \phi_0(s)ds$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s)ds$

⋮

Let $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s)ds$

Let $\phi(t) = \lim_{n\to\infty} \phi_n(t)$

Some questions:

1.) Does $\phi_n(t)$ exist for all n?

2.) Does sequence ϕ_n converge?

3.) Is $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ a solution to (*).

4.) Is the solution unique.

Example: $y' = t + 2y$. That is $f(t,y) = t + 2y$

Let $\phi_0(t) = 0$

Let $\phi_1(t) = \int_0^t f(s,0)ds = \int_0^t (s + 2(0))ds$

$$= \int_0^t s\,ds = \frac{s^2}{2}\Big|_0^t = \frac{t^2}{2}$$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds = \int_0^t f(s, \frac{s^2}{2})ds$

$$= \int_0^t (s + 2(\frac{s^2}{2}))ds = \frac{t^2}{2} + \frac{t^3}{3}$$

Let $\phi_3(t) = \int_0^t f(s, \phi_2(s))ds = \int_0^t f(s, \frac{s^2}{2} + \frac{s^3}{3})ds$

$$= \int_0^t (s + 2\frac{s^2}{2} + \frac{s^3}{3}))ds = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$$

⋮

See class notes.

$mu''(t) + \gamma u'(t) + ku(t) = 0, \quad m, \gamma, k \geq 0$

$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$

$\gamma^2 - 4km > 0$: $u(t) = Ae^{r_1 t} + Be^{r_2 t}$

$\gamma^2 - 4km = 0$: $u(t) = (A + Bt)e^{r_1 t}$

$$\gamma^2 - 4km < 0: \; u(t) = e^{-\frac{\gamma t}{2m}}(A cos \mu t + B sin \mu t) = e^{-\frac{\gamma t}{2m}} R cos(\mu t - \delta)$$

where $A = R cos(\delta)$, $B = R sin(\delta)$

μ = quasi frequency, $\frac{2\pi}{\mu}$ = quasi period

Note if $\gamma = 0$, then

Critical damping: $\gamma = 2\sqrt{km}$

Overdamped: $\gamma > 2\sqrt{km}$

Suppose a mass weighs 64 lbs stretches a spring 4 ft. If there is no damping and the spring is stretched an additional foot and set in motion with an upward velocity of $\sqrt{8}$ ft/sec, find the equation of motion of the mass.

$Weight = mg$: $m = \frac{weight}{g} = \frac{64}{32} = 2$

$mg - kL = 0$ implies $k = \frac{mg}{L} = \frac{64}{4} = 16$

$mu''(t) + \gamma u'(t) + ku(t) = F_{external}$

$[\gamma^2 - 4km < 0$: $u(t) = e^{-\frac{\gamma t}{2m}}(A cos \mu t + B sin \mu t)$
Hence $u(t) = A cos \mu t + B sin \mu t$ since $\gamma = 0]$.

$2u''(t) + 16u(t) = 0$

$u''(t) + 8u(t) = 0, \qquad u(0) = 1,\ u'(0) = -\sqrt{8}$

$r^2 + 8 = 0 \rightarrow r = \pm\sqrt{-8} = \pm i\sqrt{8} = 0 \pm i\sqrt{8}$

$u(t) = c_1 e^{it\sqrt{8}} + c_2 e^{-it\sqrt{8}}$

$u(t) = A cos\sqrt{8}t + B sin\sqrt{8}t$

$u(0) = 1$: $1 = A cos(0) + B sin(0) = A$

$u'(t) = -\sqrt{8} A sin\sqrt{8}t + \sqrt{8} B cos\sqrt{8}t$

$u'(0) = -\sqrt{8} : -\sqrt{8} = -\sqrt{8} A sin(0) + \sqrt{8} B cos(0)$

$B = -1$ Thus $u(t) = cos\sqrt{8}t - sin\sqrt{8}t$

$\Rightarrow u(t) = R cos(\mu t - \delta)$

3.7: Mechanical and Electrical Vibrations

Trig background:

$cos(y \mp x) = cos(x \mp y) = cos(x)cos(y) \pm sin(x)sin(y)$

Let $A = Rcos(\delta)$, $B = Rsin(\delta)$ in

$Acos(\omega_0 t) + Bsin(\omega_0 t)$
$= Rcos(\delta)cos(\omega_0 t) + Rsin(\delta)sin(\omega_0 t)$
$= Rcos(\omega_0 t - \delta)$

Amplitude $= R$
frequency $= \omega_0$ (measured in radians per unit time).
period $= \frac{2\pi}{\omega_0}$
phase (displacement) $= \delta$

Mechanical Vibrations:

$$mu''(t) + \gamma u'(t) + ku(t) = F_{external}, \quad m, \gamma, k \geq 0$$
$$mg - kL = 0, \qquad F_{damping}(t) = -\gamma u'(t)$$

$m=$ mass,

$k =$ spring force proportionality constant,

$\gamma =$ damping force proportionality constant

g = 9.8 m/sec

Electrical Vibrations:

$$L\frac{dI(t)}{dt} + RI(t) + \frac{1}{C}Q(t) = E(t), \quad L, R, C \geq 0 \text{ and } I = \frac{dQ}{dt}$$

$$lQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t)$$

$L =$ inductance (henrys),

$R =$ resistance (ohms)

$C =$ capacitance (farads)

$Q(t) =$ charge at time t (coulombs)

$I(t) =$ current at time t (amperes)

$E(t) =$ impressed voltage (volts).

1 volt = 1 ohm · 1 ampere = 1 coulomb / 1 farad = 1 henry · 1 amperes/ 1 second

Suppose salty water enters and leaves a tank at a rate of 2 liters/minute.

Suppose also that the salt concentration of the water entering the tank varies with respect to time according to $Q(t) \cdot tsin(t^2)$ g/liters where $Q(t)$ = amount of salt in tank in grams. (Note: this is not realistic).

If the tank contains 4 liters of water and initially contains 5g of salt, find a formula for the amount of salt in the tank after t minutes.

Let $Q(t)$ = amount of salt in tank in grams.

Note $Q(0) = 5$ g

$$\text{rate in} = (2 \text{ liters/min})(Q(t) \cdot tsin(t^2) \text{ g/liters}) = 2Qtsin(t^2) \text{ g/min}$$

$$\text{rate out} = (2 \text{ liters/min})\left(\frac{Q(t)g}{4\text{liters}}\right) = \frac{Q}{2} \text{ g/min}$$

$$\frac{dQ}{dt} = \text{rate in - rate out} = 2Qtsin(t^2) - \frac{Q}{2}$$

$$\frac{dQ}{dt} = Q(2tsin(t^2) - \tfrac{1}{2})$$

This is a first order linear ODE. It is also a separable ODE. Thus can use either 2.1 or 2.2 methods.

Using the easier 2.2:

$$\int \frac{dQ}{Q} = \int (2tsin(t^2) - \tfrac{1}{2})dt = \int 2tsin(t^2)dt - \int \tfrac{1}{2}dt$$

Let $u = t^2$, $du = 2tdt$

$$ln|Q| = \int sin(u)du - \tfrac{t}{2} = -cos(u) - \tfrac{t}{2} + C = -cos(t^2) - \tfrac{t}{2} + C$$

$$|Q| = e^{-cos(t^2) - \frac{t}{2} + C} = e^C e^{-cos(t^2) - \frac{t}{2}}$$

$$Q = Ce^{-cos(t^2) - \frac{t}{2}}$$

$Q(0) = 5: \quad 5 = Ce^{-1-0} = Ce^{-1}$. Thus $C = 5e$

Thus $Q(t) = 5e \cdot e^{-cos(t^2) - \frac{t}{2}}$

Thus $Q(t) = 5e^{-cos(t^2) - \frac{t}{2} + 1}$

Long-term behaviour:

$$Q(t) = 5(e^{-cos(t^2)})(e^{\frac{-t}{2}})e$$

As $t \to \infty$, $e^{\frac{-t}{2}} \to 0$, while $5(e^{-cos(t^2)})e$ are finite.

Thus as $t \to \infty$, $Q(t) \to 0$.

The LaPlace Transform is a method to change a differential equation to a linear equation.

Example: Solve $y'' + 3y' + 4y = 0$, $y(0) = 5$, $y'(0) = 6$

1.) Take the LaPlace Transform of both sides of the equation:

$\mathcal{L}(y'' + 3y' + 4y) = \mathcal{L}(0)$

2.) Use the fact that the LaPlace Transform is linear:

$\mathcal{L}(y'') + 3\mathcal{L}(y') + 4\mathcal{L}(y) = 0$

3.) Use thm to change this equation into an algebraic equation:

$s^2\mathcal{L}(y) - sy(0) - y'(0) + 3[s\mathcal{L}(y) - y(0)] + 4\mathcal{L}(y) = 0$

3.5) Substitute in the initial values:

$s^2\mathcal{L}(y) - 5s - 6 + 3[s\mathcal{L}(y) - 5] + 4\mathcal{L}(y) = 0$

4.) Solve the algebraic equation for $\mathcal{L}(y)$

$s^2\mathcal{L}(y) - 5s - 6 + 3s\mathcal{L}(y) - 15 + 4\mathcal{L}(y) = 0$

$[s^2 + 3s + 4]\mathcal{L}(y) = 5s + 21$

$\mathcal{L}(y) = \frac{5s+21}{s^2+3s+4}$

Some algebra implies $\mathcal{L}(y) = \frac{5s+21}{s^2+3s+4}$

5.) Solve for y by taking the inverse LaPlace transform of both sides (use a table):

$\mathcal{L}^{-1}(\mathcal{L}(y)) = \mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4})$

$y = \mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4})$

Find the inverse LaPlace transform of $\frac{5s+21}{s^2+3s+4}$

Look at the denominator first to determine if it is of the form $s^2 \pm a^2$ or $(s-a)^{n+1}$ or $(s-a)^2 + b^2$ OR if you should factor and use partial fractions

$s^2 + 3s + 4$: $b^2 - 4ac = 3^2 - 4(1)(4) = 9 - 16 < 0$

Hence s^2+3s+4 does not factor over the reals. Hence to avoid complex numbers, we won't factor it.

s^2+3s+4 is not an s^2-a^2 or an s^2+a^2 or an $(s-a)^2$, so it must be an $(s-a)^2 + b^2$.

Hence we will complete the square:

$s^2 + 3s + \_\_\_ - \_\_\_ + 4 = (s + \_\_\_)^2 - \_\_\_ + 4$

Hence $\frac{5s+21}{s^2+3s+4} = \frac{5s+21}{(s+\frac{3}{2})^2+\frac{7}{4}}$

Must now consider the numerator. We need it to look like $s - a = s + \frac{3}{2}$ or $b = \sqrt{\frac{7}{4}}$ in order to use $\mathcal{L}^{-1}(\frac{s-a}{(s-a)^2+b^2}) = e^{at}cosbt$
and/or $\mathcal{L}^{-1}(\frac{b}{(s-a)^2+b^2}) = e^{at}sinbt$

$5s + 21 = 5(s + \frac{3}{2}) - \frac{15}{2} + 21 = 5(s + \frac{3}{2}) - \frac{27}{2}$

$= 5(s + \frac{3}{2}) - [\frac{27}{2}\sqrt{\frac{4}{7}}]\sqrt{\frac{7}{4}} = 5(s + \frac{3}{2}) - [\frac{27}{\sqrt{7}}]\sqrt{\frac{7}{4}}$

Hence $\frac{5s+21}{s^2+3s+4} = \frac{5(s+\frac{3}{2}) - [\frac{27}{\sqrt{7}}]\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}$

$= 5[\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}] - \frac{27}{\sqrt{7}}[\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}]$

Thus $\mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4}) = \mathcal{L}^{-1}(5[\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}] - \frac{27}{\sqrt{7}}[\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}])$

$= 5\mathcal{L}^{-1}(\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}) - \frac{27}{\sqrt{7}}\mathcal{L}^{-1}(\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}})$

$= 5e^{-\frac{3}{2}t}cos\sqrt{\frac{7}{4}}t - \frac{27}{\sqrt{7}}e^{-\frac{3}{2}t}sin\sqrt{\frac{7}{4}}t$

Hence $y(t) = 5e^{-\frac{3}{2}t}cos\sqrt{\frac{7}{4}}t - \frac{27}{\sqrt{7}}e^{-\frac{3}{2}t}sin\sqrt{\frac{7}{4}}t$.

$$g(t) = \begin{cases} 0 & t < 4 \\ 2 & 4 \le t < 10 \\ t & t \ge 10 \end{cases}$$

Hence $g(t) = 2u_4(t) + (t-2)u_{10}(t)$

Solve $3y'' + y' + y = 2u_4(t) + (t-2)u_{10}(t)$,
$y(0) = 0,\ y'(0) = 0.$

$$3\mathcal{L}(y'') + \mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(2u_4(t)) + \mathcal{L}((t-2)u_{10}(t))$$

Thm: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$.

Thus $\mathcal{L}(u_c(t)f(t)) =$ ________________

$$3[s^2\mathcal{L}(y) - sy(0) - y'(0)] + s\mathcal{L}(y) - y(0) + \mathcal{L}(y) = e^{-4s}\mathcal{L}(2) + e^{-10s}\mathcal{L}((t+8))$$

$$3[s^2\mathcal{L}(y)] + s\mathcal{L}(y) + \mathcal{L}(y) = 2e^{-4s}\mathcal{L}(1) + e^{-10s}\mathcal{L}(t) + 8e^{-10s}\mathcal{L}(1)$$

$$\mathcal{L}(y)[3s^2 + s + 1] = e^{-4s}\tfrac{2}{s} + e^{-10s}\tfrac{1}{s^2} + e^{-10s}\tfrac{8}{s}$$

$$\mathcal{L}(y) = e^{-4s}\frac{2}{s[3s^2+s+1]} + e^{-10s}\frac{1}{s^2[3s^2+s+1]} + 8e^{-10s}\frac{1}{s[3s^2+s+1]}$$

$$y = 2\mathcal{L}^{-1}\left(e^{-4s}\frac{1}{s[3s^2+s+1]}\right) + \mathcal{L}^{-1}\left(e^{-10s}\frac{1}{s^2[3s^2+s+1]}\right) + 8\mathcal{L}^{-1}\left(e^{-10s}\frac{1}{s[3s^2+s+1]}\right)$$

$$y = u_4(t)f(t-4) + u_{10}h(t-10) + 8u_{10}f(t-10)$$

where $f(t) = \mathcal{L}^{-1}\left(\frac{1}{s[3s^2+s+1]}\right)$ and $h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$

$$\frac{1}{s[3s^2+s+1]} = \frac{A}{s} + \frac{Bs+C}{3s^2+s+2}$$

$$1 = A(3s^2+s+1) + (Bs+C)s$$

$$0s^2 + 0s + 1 = (3A+B)s^2 + (A+C)s + A$$

$$0 = 3A + B,\ 0 = A + C,\ 1 = A$$

Hence $A = 1,\ B = -3A = -3,\ C = -A = -1$

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s[3s^2+s+1]}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{-3s-1}{3s^2+s+1}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{-3s-1}{3s^2+s+1}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3[s^2+\frac{1}{3}s+\frac{1}{3}]}\right))$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3[(s^2+\frac{1}{3}s+\_\_\_\_) - \_\_\_\_ + \frac{1}{3}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3[(s+\frac{1}{6})^2 - \frac{1}{36} + \frac{1}{3}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3(s+\frac{1}{3})}{3[(s+\frac{1}{6})^2 + \frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6}-\frac{1}{6}+\frac{1}{3})}{[(s+\frac{1}{6})^2 + \frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6}+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{6}\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{\sqrt{11}}\frac{\sqrt{11}}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

Thm: $\mathcal{L}^{-1}(F(s-c)) = e^{ct}\mathcal{L}^{-1}(F(s))$

$$= 1 + e^{-\frac{1}{6}t}\mathcal{L}^{-1}\left(\frac{-s}{[s^2+\frac{11}{36}]}\right) - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}t}\mathcal{L}^{-1}\left(\frac{\frac{\sqrt{11}}{6}}{[s^2+\frac{11}{36}]}\right)$$

$$= 1 - e^{-\frac{1}{6}t}cos\frac{\sqrt{11}}{6}t - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}t}sin\frac{\sqrt{11}}{6}t$$

$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$

$\frac{1}{s^2[3s^2+s+1]} = \frac{As+D}{s^2} + \frac{Bs+C}{3s^2+s+2}$

$1 = (As+D)(3s^2+s+1) + (Bs+C)s^2$

$0s^3 + 0s^2 + 0s + 1 = (3A+B)s^3 + (A+3D+C)s^2 + (A+D)s + D$

$0 = 3A+B,\ 0 = A+3D+C,\ 0 = A+D,\ 1 = D.$

Hence $D = 1$, $A = -D = -1$, $C = -A - 3D = 1 - 3 = -2$, $B = -3A = 3$.

$$\frac{1}{s^2[3s^2+s+1]} = \frac{-s+1}{s^2} + \frac{3s-2}{3s^2+s+1} = \frac{-s}{s^2} + \frac{1}{s^2} + \frac{3(s-\frac{2}{3})}{3[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6}-\frac{1}{6}-\frac{2}{3})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} = \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6}-\frac{5}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{(\frac{5}{6})(\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{(\frac{5}{6})(\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{\frac{5}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$

$$= \mathcal{L}^{-1}\left(\frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{\frac{5}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= -1 + t + e^{-\frac{1}{6}t}cos\frac{\sqrt{11}}{6}t - \frac{5}{\sqrt{11}}e^{-\frac{1}{6}t}sin\frac{\sqrt{11}}{6}t$$

Hence the final answer is

$y = u_4(t)f(t-4) + u_{10}h(t-10) + 8u_{10}f(t-10)$

$= u_4(t)[1 - e^{-\frac{1}{6}(t-4)}cos\frac{\sqrt{11}}{6}(t-4) - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}(t-4)}sin\frac{\sqrt{11}}{6}(t-4)] +$
$u_{10}[-1+t-10+e^{-\frac{1}{6}(t-10)}cos\frac{\sqrt{11}}{6}(t-10) - \frac{5}{\sqrt{11}}e^{-\frac{1}{6}(t-10)}sin\frac{\sqrt{11}}{6}(t-10)]$ [redacted]
$+8u_{10}[1 - e^{-\frac{1}{6}(t-10)}cos\frac{\sqrt{11}}{6}(t-10) - \frac{1}{\sqrt{11}}e^{-\frac{1}{6}(t-10)}sin\frac{\sqrt{11}}{6}(t-10)]$

Section 6.3

Example: $f(t) = \begin{cases} f_1, & \text{if } t < 4; \\ f_2, & \text{if } 4 \leq t < 5; \\ f_3, & \text{if } 5 \leq t < 10; \\ f_4, & \text{if } t \geq 10; \end{cases}$

Hence $f(t) = f_1(t) + u_4(t)[f_2(t) - f_1(t)] + u_5(t)[f_3(t) - f_2(t)] + u_{10}(t)[f_4(t) - f_3(t)]$

Formula 13: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$.

or equivalently

$$\mathcal{L}(u_c(t)f(t-c+c)) = e^{-cs}\mathcal{L}(f(t+c)).$$

or equivalently

$$\mathcal{L}(u_c(t)f(t)) = e^{-cs}\mathcal{L}(f(t+c)).$$

In other words, replacing $t-c$ with t is equivalent to replacing t with $t+c$

Formula 13: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$.

Let $F(s) = \mathcal{L}(f(t))$. Then $\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}(\mathcal{L}(f(t))) = f(t)$.

Thus $\mathcal{L}^{-1}(e^{-cs}F(s)) = \mathcal{L}^{-1}(e^{-cs}\mathcal{L}(f(t))) = u_c(t)f(t-c)$ where $f(t) = \mathcal{L}^{-1}(F(s))$ ∎

6.5: Impulse functions

Unit impulse function = Dirac delta function is a generalized function with the properties

$$\delta(t) = 0, \quad t \neq 0 \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\mathcal{L}(\delta(t - t_0)) = e^{-st_0}$$

← on formula sheet

Let $d_k(t) = \begin{cases} \frac{1}{2k} & -k < t < k \\ 0 & t \leq -k \text{ or } t \geq k \end{cases}$

Note $lim_{k \to 0} d_k(t) = 0$ if $t \neq 0$

and $lim_{k \to 0} \int_{-\infty}^{\infty} d_k(t) = lim_{k \to 0} 1 = 1 = \int_{-\infty}^{\infty} \delta(t)dt$

$$\mathcal{L}(\delta(t - t_0)) = lim_{k \to 0} \mathcal{L}(d_k(t - t_0))$$

$$= lim_{k \to 0} \int_0^{\infty} e^{-st} d_k(t - t_0)dt$$

$$= lim_{k \to 0} \frac{1}{2k} \int_{t_0 - k}^{t_0 + k} e^{-st}dt$$

$$= lim_{k \to 0} \frac{-1}{2sk} e^{-st} \Big|_{t_0 - k}^{t_0 + k}$$

$$= lim_{k \to 0} \frac{1}{2sk} e^{-st_0}(e^{sk} - e^{-sk})$$

$$= lim_{k \to 0} \frac{sinh(sk)}{sk} e^{-st_0}$$

$$= lim_{k \to 0} \frac{scosh(sk)}{s} e^{-st_0} \quad = \quad e^{-st_0}$$

$sin(t) = \frac{e^{it} - e^{-it}}{2i}$ $\qquad cos(t) = \frac{e^{it} + e^{-it}}{2}$

$sinh(t) = \frac{e^{t} - e^{-t}}{2}$ $\qquad cosh(t) = \frac{e^{t} + e^{-t}}{2}$

$[sinh(t)]' =$ $\qquad [cosh(t)]' =$

$sinh(0) = \frac{e^{0} - e^{0}}{2} = 0$ $\qquad cosh(0) = \frac{e^{0} + e^{0}}{2} = 1$

Intro to Group Theory

Define the $\cdot$ product on R^2 by

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 - y_1x_2)$$

Note $\cdot$ is

1.) commutative:
$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 - y_1x_2)$
$= (x_2x_1 - y_2y_1, x_2y_1 - y_2x_1) = (x_2, y_2) \cdot (x_1, y_1)$

2.) associative: $(f \cdot g) \cdot h = f \cdot (g \cdot h)$

3.) distributive w.r.t +: $f \cdot (g_1 + g_2) = f \cdot g_1 + f \cdot g_2$

4.) $(x_1, y_1) \cdot (0, 0) = (0, 0)$

Note $(0, 1) \cdot (0, 1) = (-1, 0)$

6.6: The Convolution Integral

Defn: The convolution of f and g is the function $f*g$ defined by

$(f*g)(t) = \int_0^t f(t-s)g(s)ds = \int_0^t f(x)g(t-x)dx$

Note $*$ is

1.) commutative: $f*g = g*f$

2.) associative: $(f*g)*h = f*(g*h)$

3.) distributive w.r.t +: $f*(g_1+g_2) = f*g_1 + f*g_2$

4.) $f*0 = 0*f = 0$

Example: $cos(t)*1 =$

Example: $t*t \not\geq 0$

Thm: $\mathcal{L}((f*g)(t)) = \mathcal{L}(f(t)) \cdot \mathcal{L}(g(t))$

Proof:

$\mathcal{L}(f(t))\mathcal{L}(g(t)) = \int_0^\infty e^{-sy}f(y)dy \int_0^\infty e^{-sx}g(x)dx$

$= \int_0^\infty [\int_0^\infty e^{-sy}f(y)dy]e^{-sx}g(x)dx$

$= \int_0^\infty [\int_0^\infty e^{-sy}f(y)e^{-sx}g(x)dy]dx$

$= \int_0^\infty [\int_0^\infty e^{-s(y+x)}f(y)g(x)dy]dx$

$= \int_0^\infty [\int_0^\infty e^{-s(y+x)}f(y)g(x)dx]dy$

Let $t = x+y$, $dt = dx$

$= \int_0^\infty [\int_y^\infty e^{-st}f(y)g(t-y)dt]dy$

$= \int_0^\infty [\int_0^t e^{-st}f(y)g(t-y)dy]dt$

$= \int_0^\infty e^{-st}[\int_0^t f(y)g(t-y)dy]dt$

$= \int_0^\infty e^{-st}(f*g)(t)dt$

$= \mathcal{L}(f*g)$

Example: $\mathcal{L}^{-1}(\frac{1}{s(s-a)}) =$

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
4. $t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 7
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 7
9. $e^{at} \sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}, \quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t) f(t-c)$	$e^{-cs} F(s)$	Sec. 6.3
14. $e^{ct} f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c} F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 25
16. $\int_0^t f(t-\tau) g(\tau)\, d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2; Cor. 6.2.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 29

$x' = A\vec{x}$

Solve $\mathbf{X}'(t) = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X(t)}$

Step 1. Find eigenvalues:

$$A - \lambda I = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda) - 12$$
$$= \lambda^2 - 6\lambda + 5 - 12 = \lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1) = 0$$

Thus $\lambda = 7, -1$

Step 2. Find eigenvectors:

$\lambda = 7$: $\quad A - 7I = \begin{bmatrix} 1-7 & 3 \\ 4 & 5-7 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$

Note $\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\qquad \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Note the dimension of the nullspace of $\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$ is 1.

Or in other words, solution space for

$$\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ is 1-dimensional}$$

Thus a basis for the eigenspace for $\lambda = 7$ is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an e. vector for $\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = A$

$A\vec{v} = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ✓

$\lambda = -1$ $\quad A - (-1)I = \begin{bmatrix} 1+1 & 3 \\ 4 & 5+1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

Note $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Thus a basis for the eigenspace for $\lambda = -1$ is $\left\{ \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$

Thus a basis for the solution space to $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} \right\}$$

Hence the general solution is

$$\mathbf{X}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t}$$

Note we can take any basis for the solution space to create the general solution

Alternate basis: $\left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{7t}, \begin{bmatrix} -9 \\ 6 \end{bmatrix} e^{-t} \right\}$

Alternate format of general solution:

$$\mathbf{X}(t) = c_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} -9 \\ 6 \end{bmatrix} e^{-t}$$

part b #7.7

IVP: $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}, \quad \mathbf{X}(t_0) = \begin{bmatrix} e \\ f \end{bmatrix}$

$$\begin{bmatrix} e \\ f \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t_0} + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t_0} = \begin{bmatrix} c_1 e^{7t_0} + 3c_2 e^{-t_0} \\ 2c_1 e^{7t_0} - 2c_2 e^{-t_0} \end{bmatrix}$$

Solve using any method you like. We will use matrix form:

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Solution exists if Wronskian evaluated at t_0 is not zero.

$$W\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} \right) = \begin{vmatrix} e^{7t} & 3e^{-t} \\ 2e^{7t} & -2e^{-t} \end{vmatrix}$$

$$= -2e^{6t} - 6e^{6t} = -8e^{6t} \neq 0$$

Section 7.7

Fundamental matrix: $\Phi(t) = \begin{bmatrix} e^{7t} & 3e^{-t} \\ 2e^{7t} & -2e^{-t} \end{bmatrix}$

Back to IVP: $\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix}^{-1} \begin{bmatrix} e \\ f \end{bmatrix}$$

$$= \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix}^{-1} \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix}^{-1} \begin{bmatrix} e \\ f \end{bmatrix}$

Extra

But I would prefer a fundamental matrix whose inverse is easier to calculate, at least when $t_0 = 0$.

Thus we will find another basis for the solution set to $\mathbf{x}' = A\mathbf{x}$ so that the corresponding fundamental matrix has the property that $\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the 2x2 identity matrix.

Step 1: Solve IVP: $\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ← part b #7.7

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^0 = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \text{ implies } \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(-\tfrac{1}{8}) \begin{bmatrix} -2 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix} \;\&\; \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Thus IVP solution where $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is

$$\mathbf{X}(t) = \tfrac{1}{4} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t} + \tfrac{1}{4} \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} \end{bmatrix}$$

Solve to IVP
$Y' = AY, \; Y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
A soln

part 6 7 7

Step 2: Solve IVP: $\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^0 = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ -\frac{1}{8} \end{bmatrix}$

Thus IVP solution where $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is

$$\mathbf{X}(t) = \frac{3}{8} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t} - \frac{1}{8} \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} = \begin{bmatrix} \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix}$$

2nd soln to IVP $x' = Ax$, $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Thus another basis for the solution space to $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}$

is $\left\{ \begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} \end{bmatrix}, \begin{bmatrix} \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix} \right\}$

Its corresponding fundamental matrix is

$$\begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} & \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} & \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix}$$

General sol

$$\vec{X} = c_1 \begin{bmatrix} e^{7t}/4 + 3e^{-t}/4 \\ e^{7t}/2 - e^{-t}/2 \end{bmatrix} + c_2 \begin{bmatrix} \frac{3e^{7t}}{8} - \frac{3e^{-t}}{8} \\ \frac{3e^{7t}}{4} + \frac{e^{-t}}{4} \end{bmatrix}$$

Thus to solve IVP where $\mathbf{X}(t_0) = \begin{bmatrix} e \\ f \end{bmatrix}$, we solve

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} \frac{1}{4}e^{7t_0} + \frac{3}{4}e^{-t_0} & \frac{3}{8}e^{7t_0} - \frac{3}{8}e^{-t_0} \\ \frac{1}{2}e^{7t_0} - \frac{1}{2}e^{-t_0} & \frac{3}{4}e^{7t_0} + \frac{1}{4}e^{-t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

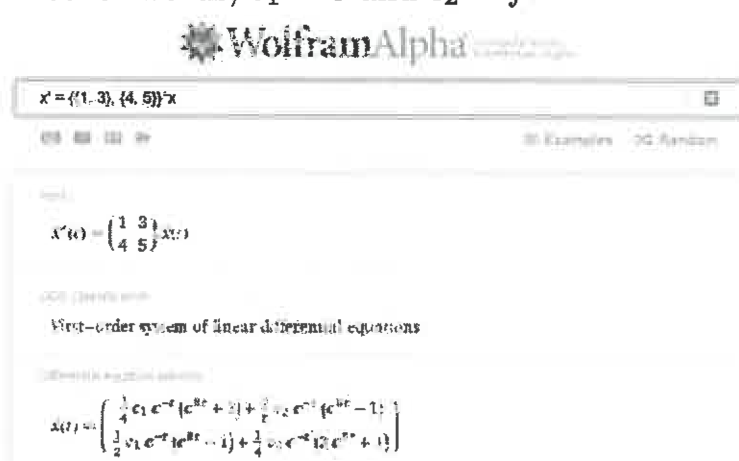
When $t_0 = 0$. I.e., we have an IVP where $\mathbf{X}(0) = \begin{bmatrix} e \\ f \end{bmatrix}$

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} \frac{1}{4}e^{0} + \frac{3}{4}e^{0} & \frac{3}{8}e^{0} - \frac{3}{8}e^{0} \\ \frac{1}{2}e^{0} - \frac{1}{2}e^{0} & \frac{3}{4}e^{0} + \frac{1}{4}e^{0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

In other words, $c_1 = e$ and $c_2 = f$.



Wolfram Alpha's soln

Ch 7 and 9

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by

$$x_1'(t) = ax_1 + bx_2,$$
$$x_2'(t) = cx_1 + dx_2$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc = 0.$$

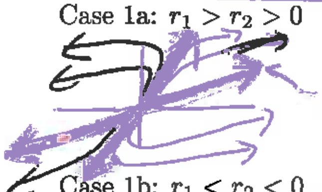
Thus $r = \frac{(a+d)\pm\sqrt{(a+d)^2-4(ad-bc)}}{2}$

Case 1: $(a+d)^2 - 4(ad-bc) > 0$ two real e. values

Hence the general solutions is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{r_2 t}$

$x_2 = \frac{v_2}{v_1} x_1$

Case 1a: $r_1 > r_2 > 0$ nodal source



Case 1b: $r_1 < r_2 < 0$ nodal sink

Case 1c: $r_2 < 0 < r_1$ saddle

Case 2: $(a+d)^2 - 4(ad-bc) = 0$ 1 repeated

Case 2i: Two independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{rt}$

Case 2ii: One independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} t + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] e^{rt}$

Case 2a: $r > 0$

Case 2b: $r < 0$

Case 3: $(a+d)^2 - 4(ad-bc) < 0$. I.e., $r = \lambda \pm i\mu$ 2 complex e. value

Suppose the eigenvector corresponding to this eigenvalue is

$$\begin{pmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Then general solution is $\vec{x} = c_1(\vec{v}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 cos(\mu t) - w_1 sin(\mu t) \\ v_2 cos(\mu t) - w_2 sin(\mu t) \end{pmatrix} e^{\lambda t} + c_2 \begin{pmatrix} v_1 sin(\mu t) + w_1 cos(\mu t) \\ v_2 sin(\mu t) + w_2 cos(\mu t) \end{pmatrix} e^{\lambda t}$$

Case 3a: $\lambda > 0$ spiral source

Case 3a: $\lambda < 0$ spiral sink

Case 3a: $\lambda = 0$ center

Solve : $\vec{x}' = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}$ has e.vectors $c_1 \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ w/ e.value = -1

and e.vectors $c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ w/ e.value = 5

Thus general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}$$

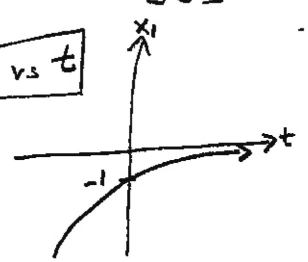
I.V.P. : Suppose $\vec{x}(0) = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \vec{x}(0) = c_1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^0$$

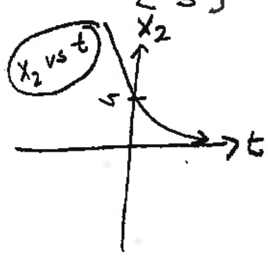
$$\left.\begin{aligned} -1 &= -c_1 + c_2 \\ 5 &= 5c_1 + c_2 \end{aligned}\right\} \Rightarrow c_1 = 1, \; c_2 = 0$$

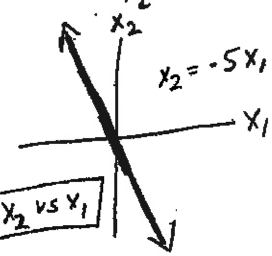
If $\vec{x}(0) = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} e^{-t} \Rightarrow \begin{aligned} x_1 &= -e^{-t} \\ x_2 &= 5e^{-t} \end{aligned}$

x_1 vs t



x_2 vs t





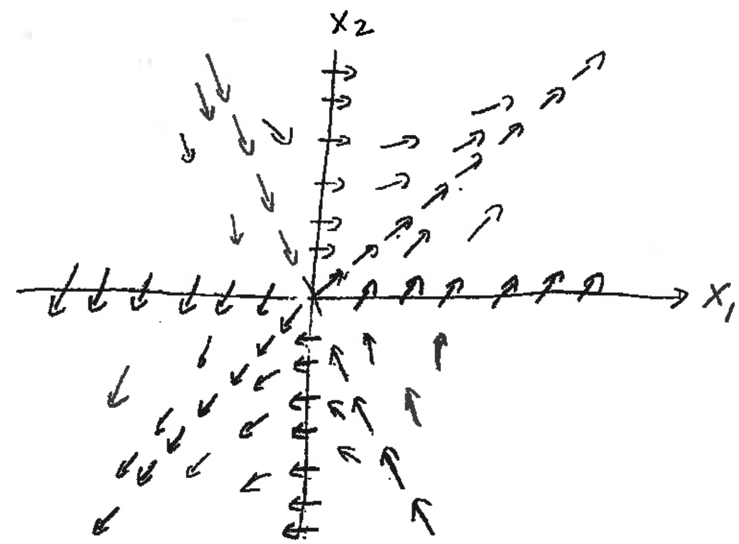
x_2 vs x_1

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4x_1 + x_2 \\ 5x_1 \end{bmatrix}$$

$\frac{dx_1}{dt} = 4x_1 + x_2$

$\frac{dx_2}{dt} = 5x_1$

$$\frac{dx_2}{dx_1} = \frac{dx_2}{dt} \cdot \frac{dt}{dx_1} = \frac{dx_2/dt}{dx_1/dt} = \frac{x_2'}{x_1'} = \boxed{\frac{5x_1}{4x_1 + x_2}}$$



If $x_2 = -5x_1 \Rightarrow \frac{x_2'}{x_1'} = \frac{5x_1}{4x_1 - 5x_1} = \frac{5x_1}{-x_1} = -5$