$\vec{x}' = A\vec{x}$

Solve $\mathbf{X}'(t) = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}(t)$

Step 1. Find eigenvalues:

$$A - \lambda I = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda) - 12$$
$$= \lambda^2 - 6\lambda + 5 - 12 = \lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1) = 0$$

Thus $\lambda = 7, -1$

Step 2. Find eigenvectors:

$\lambda = 7$: $\quad A - 7I = \begin{bmatrix} 1-7 & 3 \\ 4 & 5-7 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$

Note $\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Note the dimension of the nullspace of $\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$ is 1.

Or in other words, solution space for

$$\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ is 1-dimensional}$$

Thus a basis for the eigenspace for $\lambda = 7$ is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an e.vector $\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = A$

$A\vec{v} = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ✓

$\lambda = -1 \quad A - (-1)I = \begin{bmatrix} 1+1 & 3 \\ 4 & 5+1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

Note $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Thus a basis for the eigenspace for $\lambda = -1$ is $\left\{ \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$

Thus a basis for the solution space to $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t}, \quad \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} \right\}$$

Hence the general solution is

$$\boxed{\mathbf{X}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t}}$$

Note we can take any basis for the solution space to create the general solution

Alternate basis: $\left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{7t}, \quad \begin{bmatrix} -9 \\ 6 \end{bmatrix} e^{-t} \right\}$

Alternate format of general solution:

$$\mathbf{X}(t) = c_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} -9 \\ 6 \end{bmatrix} e^{-t}$$

IVP: $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}, \quad \mathbf{X}(t_0) = \begin{bmatrix} e \\ f \end{bmatrix}$

$$\begin{bmatrix} e \\ f \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t_0} + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t_0} = \begin{bmatrix} c_1 e^{7t_0} + 3c_2 e^{-t_0} \\ 2c_1 e^{7t_0} - 2c_2 e^{-t_0} \end{bmatrix}$$

Solve using any method you like. We will use matrix form:

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Solution exists if Wronskian evaluated at t_0 is not zero.

$$W\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t}\right) = \begin{vmatrix} e^{7t} & 3e^{-t} \\ 2e^{7t} & -2e^{-t} \end{vmatrix}$$

$$= -2e^{6t} - 6e^{6t} = -8e^{6t} \neq 0$$

section 7.7

Fundamental matrix: $\Phi(t) = \begin{bmatrix} e^{7t} & 3e^{-t} \\ 2e^{7t} & -2e^{-t} \end{bmatrix}$

Back to IVP: $\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix}^{-1} \begin{bmatrix} e \\ f \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix}^{-1} \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e^{7t_0} & 3e^{-t_0} \\ 2e^{7t_0} & -2e^{-t_0} \end{bmatrix}^{-1} \begin{bmatrix} e \\ f \end{bmatrix}$

part b of 7.7

But I would prefer a fundamental matrix whose inverse is easier to calculate, at least when $t_0 = 0$.

Thus we will find another basis for the solution set to $\mathbf{x}' = A\mathbf{x}$ so that the corresponding fundamental matrix has the property that $\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the 2x2 identity matrix.

Step 1: Solve IVP: $\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ← part b #7.7

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^0 = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \text{ implies } \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(-\tfrac{1}{8}) \begin{bmatrix} -2 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix} \;\&\; \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Thus IVP solution where $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is

$$\mathbf{X}(t) = \frac{1}{4} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t} + \frac{1}{4} \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} = \boxed{\begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} \end{bmatrix}}$$

Solve to IVP
$Y' = AY, \; \bar{Y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ → A soln

part 6 & 7 7

$$\boxed{\textbf{Step 2: Solve IVP:} \quad \mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^0 = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ -\frac{1}{8} \end{bmatrix}$

Thus IVP solution where $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is

2nd sol'n to $X' = AX$, IVP $X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\mathbf{X}(t) = \frac{3}{8} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{7t} - \frac{1}{8} \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t} = \boxed{\begin{bmatrix} \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix}}$$

Thus another basis for the solution space to $\mathbf{X}' = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \mathbf{X}$

is $\left\{ \begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} \end{bmatrix}, \begin{bmatrix} \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix} \right\}$

Its corresponding fundamental matrix is

$$\begin{bmatrix} \frac{1}{4}e^{7t} + \frac{3}{4}e^{-t} & \frac{3}{8}e^{7t} - \frac{3}{8}e^{-t} \\ \frac{1}{2}e^{7t} - \frac{1}{2}e^{-t} & \frac{3}{4}e^{7t} + \frac{1}{4}e^{-t} \end{bmatrix}$$

General sol

$$\vec{X} = c_1 \begin{bmatrix} e^{7t}/4 + 3e^{-t}/4 \\ e^{7t}/2 - e^{-t}/2 \end{bmatrix} + c_2 \begin{bmatrix} \frac{3e^{7t}}{8} - \frac{3e^{-t}}{8} \\ \frac{3e^{7t}}{4} + \frac{e^{-t}}{4} \end{bmatrix}$$

Thus to solve IVP where $\mathbf{X}(t_0) = \begin{bmatrix} e \\ f \end{bmatrix}$, we solve

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} \frac{1}{4}e^{7t_0} + \frac{3}{4}e^{-t_0} & \frac{3}{8}e^{7t_0} - \frac{3}{8}e^{-t_0} \\ \frac{1}{2}e^{7t_0} - \frac{1}{2}e^{-t_0} & \frac{3}{4}e^{7t_0} + \frac{1}{4}e^{-t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

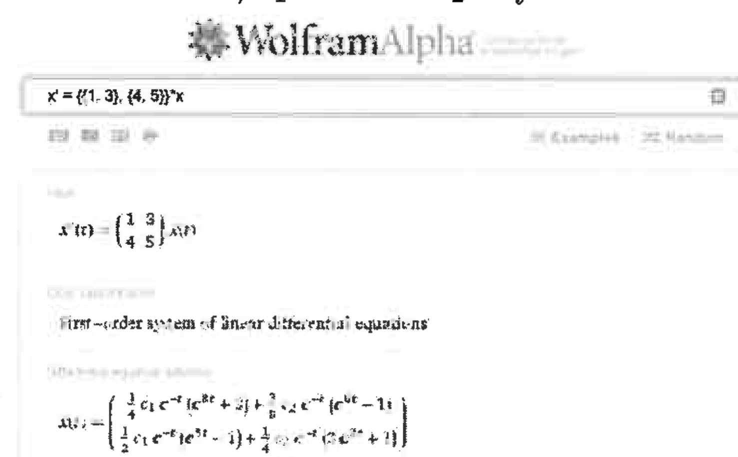
When $t_0 = 0$. I.e., we have an IVP where $\mathbf{X}(0) = \begin{bmatrix} e \\ f \end{bmatrix}$

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} \frac{1}{4}e^0 + \frac{3}{4}e^0 & \frac{3}{8}e^0 - \frac{3}{8}e^0 \\ \frac{1}{2}e^0 - \frac{1}{2}e^0 & \frac{3}{4}e^0 + \frac{1}{4}e^0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

In other words, $c_1 = e$ and $c_2 = f$.



Wolfram Alpha's sol'n

Ch 7 and 9

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by

$$x_1'(t) = ax_1 + bx_2,$$
$$x_2'(t) = cx_1 + dx_2$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

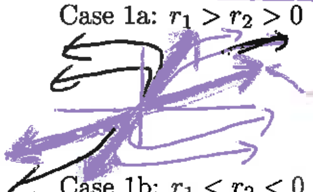
$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc = 0.$$

Thus $r = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

Case 1: $(a+d)^2 - 4(ad-bc) > 0$ two real e. values

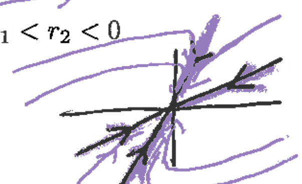
Hence the general solutions is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{r_2 t}$

Case 1a: $r_1 > r_2 > 0$



$x_2 = \frac{v_2}{v_1} x_1$ nodal source

Case 1b: $r_1 < r_2 < 0$



nodal sink

Case 1c: $r_2 < 0 < r_1$

saddle

Case 2: $(a+d)^2 - 4(ad-bc) = 0$ 1 repeated

Case 2i: Two independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{rt}$

Case 2ii: One independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} t + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] e^{rt}$

Case 2a: $r > 0$

Case 2b: $r < 0$

Case 3: $(a+d)^2 - 4(ad-bc) < 0$. I.e., $r = \lambda \pm i\mu$ 2 complex e. value

Suppose the eigenvector corresponding to this eigenvalue is

$$\begin{pmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

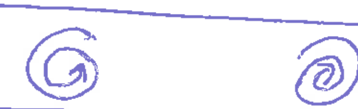
Then general solution is $\vec{X} = c_1 (\vec{V}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 cos(\mu t) - w_1 sin(\mu t) \\ v_2 cos(\mu t) - w_2 sin(\mu t) \end{pmatrix} e^{\lambda t} + c_2 \begin{pmatrix} v_1 sin(\mu t) + w_1 cos(\mu t) \\ v_2 sin(\mu t) + w_2 cos(\mu t) \end{pmatrix} e^{\lambda t}$$

Case 3a: $\lambda > 0$

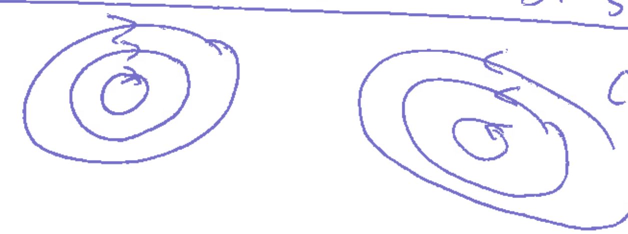
spiral sou

Case 3a: $\lambda < 0$



spiral sink

Case 3a: $\lambda = 0$



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