3.6 Variation of Parameters　　Solve $y'' - 2y' + y = e^t ln(t)$

1) Find homogeneous solutions: Solve $y'' - 2y' + y = 0$

homog

Guess: $y = e^{rt}$, then $y' = re^{rt}$, $y'' = r^2e^{rt}$, and

$$r^2e^{rt} - 2re^{rt} + e^{rt} = 0 \text{ implies } \boxed{r^2 - 2r + 1 = 0}$$

$(r-1)^2 = 0$, and hence $r = 1$

General homogeneous solution: $y = c_1e^t + c_2te^t$

since have two linearly independent solutions: $\{e^t, te^t\}$

2.) Find a non-homogeneous solution:

Sect. 3.5 method: Educated guess　$y = c_1 e^t + c_2 t e^t + Y(t)$

Sect. 3.6: **Guess $y = u_1(t)e^t + u_2(t)te^t$ and solve for u_1 and u_2**

find one nonhom

$$u_1(t) = \int \frac{\begin{vmatrix} 0 & \phi_2 \\ 1 & \phi_2' \end{vmatrix}}{\underbrace{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}}_{W(\phi_1,\phi_2)}} g(t)dt = -\int \frac{\phi_2(t)g(t)}{W(\phi_1,\phi_2)}dt = -\int \frac{(te^t)(e^t ln(t))}{e^{2t}}dt$$

$$= -\int t ln(t) = -[\frac{t^2 ln(t)}{2} - \int \frac{t}{2}] = -\frac{t^2 ln(t)}{2} + \frac{t^2}{4}$$

$$u_2(t) = \int \frac{\begin{vmatrix} \phi_1 & 0 \\ \phi_1' & 1 \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}} g(t)dt = \int \frac{\phi_1(t)g(t)}{W(\phi_1,\phi_2)}dt = \int \frac{(e^t)(e^t ln(t))}{e^{2t}}dt$$

$$= \int ln(t) = t ln(t) - t$$

$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}$$

$u = ln(t)$　$dv = tdt$
$du = \frac{dt}{t}$　$v = \frac{t^2}{2}$

$u = ln(t)$　$dv = dt$
$du = \frac{dt}{t}$　$v = t$

General solution : $y = c_1e^t + c_2te^t + (-\frac{t^2 ln(t)}{2} + \frac{t^2}{4})e^t + (t ln(t) - t)te^t$

which simplifies to $y = c_1e^t + c_2te^t + (\frac{ln(t)}{2} - \frac{3}{4})t^2e^t$

Solve $\boxed{y'' + p(t)y' + q(t)y = g(t)}$ where $y = c_1\phi_1(t) + c_2\phi_2(t)$ is solution to homogeneous equation $y'' + p(t)y' + q(t)y = 0$

Guess $y = u_1(t)\phi_1(t) + u_2(t)\phi_2(t)$

$y = u_1\phi_1 + u_2\phi_2$ implies $y' = u_1\phi_1' + u_1'\phi_1 + u_2\phi_2' + u_2'\phi_2$

Two unknown functions, u_1 and u_2, but only one equation ($y'' + p(t)y' + q(t)y = g(t)$). Thus might be OK to choose 2nd eq'n.

Avoid 2nd derivative in y'': $\boxed{\text{Choose } u_1'\phi_1 + u_2'\phi_2 = 0}$　no u_1'' no u_2''

$y' = u_1\phi_1' + u_2\phi_2'$ implies $y'' = u_1\phi_1'' + u_1'\phi_1' + u_2\phi_2'' + u_2'\phi_2'$

Plug into $y'' + p(t)y' + q(t)y = g(t)$:　2 eqns to solve for u_1 & u_2

$u_1\phi_1'' + u_1'\phi_1' + u_2\phi_2'' + u_2'\phi_2' + p(u_1\phi_1' + u_2\phi_2') + q(u_1\phi_1 + u_2\phi_2) = g$

$u_1\phi_1'' + u_1'\phi_1' + u_2\phi_2'' + u_2'\phi_2' + pu_1\phi_1' + pu_2\phi_2' + qu_1\phi_1 + qu_2\phi_2 = g$

$u_1\phi_1'' + pu_1\phi_1' + qu_1\phi_1 + u_1'\phi_1' + u_2\phi_2'' + pu_2\phi_2' + qu_2\phi_2 + u_2'\phi_2' = g$

$u_1(\underbrace{\phi_1'' + p\phi_1' + q\phi_1}_{0}) + u_1'\phi_1' + u_2(\underbrace{\phi_2'' + p\phi_2' + q\phi_2}_{0}) + u_2'\phi_2' = g$

ϕ_1, ϕ_2 are homogeneous solutions. Thus $\phi_i'' + p\phi_i' + q\phi_i = 0$.

Hence $\boxed{u_1(0) + u_1'\phi_1' + u_2(0) + u_2'\phi_2' = g}$

Thus we have 2 eqns to find 2 unknowns, the functions u_1 and u_2:

$$\begin{matrix} u_1'\phi_1 + u_2'\phi_2 = 0 \\ u_1'\phi_1' + u_2'\phi_2' = g \end{matrix} \text{ implies } \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

Cramer's rule: $u_1'(t) = \frac{\begin{vmatrix} 0 & \phi_2 \\ g & \phi_2' \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}}$ and $u_2'(t) = \frac{\begin{vmatrix} \phi_1 & 0 \\ \phi_1' & g \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}}$

$$\Rightarrow u_1 = \int \frac{\begin{vmatrix} 0 & \phi_2 \\ g & \phi_2' \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}} dt \quad \text{and } u_2 = \int \frac{||}{||} dt$$

Sect.3.6: **Guess** $y = u_1(t)e^t + u_2(t)te^t$ **and solve for** u_1 **and** u_2

$y' = u_1'e^t + u_1e^t + u_2'te^t + u_2(e^t + te^t) = e^{2t} + te^{2t} - te^{2t} - e^{2t}$.

Two unknown functions, u_1 and u_2, but only one equation ($y'' - 2y' + y = e^t ln(t)$). Thus might be OK to choose 2nd eq'n.

Avoid 2nd derivative in y'': $\boxed{\textbf{Choose } u_1'e^t + u_2'te^t = 0}$

Hence $y' = u_1e^t + u_2(e^t + te^t)$.

and $y'' = u_1'e^t + u_1e^t + u_2'(e^t + te^t) + u_2(e^t + e^t + te^t)$.

$= u_1'e^t + u_1e^t + u_2'e^t + u_2'te^t + u_2(2e^t + te^t)$.

$= u_1e^t + u_2'e^t + u_2(2e^t + te^t)$. ← no u_1'' or u_2''

Solve $y'' - 2y' + y = e^t ln(t)$

$$\boxed{u_1e^t + u_2'e^t + u_2(2e^t + te^t) - 2[u_1e^t + u_2(e^t + te^t)] + u_1e^t + u_2te^t = e^t ln(t)}$$

$u_2'e^t + 2u_2e^t + u_2te^t - 2u_2e^t - 2u_2te^t + u_2te^t = e^t ln(t)$

$u_2' = ln(t)$ or in other words, $\frac{du_2}{dt} = ln(t)$

Thus $\int du_2 = \int ln(t)dt$

$u_2 = tln(t) - t$. Note only need one solution, so don't need $+C$.

$y = u_1(t)e^t + [tln(t) - t]te^t$

$u_1'e^t + u_2'te^t = 0$. Thus $u_1' + u_2't = 0$. Hence $u_1' = -u_2't = -tln(t)$

Thus $u_1 = -\int tln(t)dt = -\frac{t^2 ln(t)}{2} + \frac{t^2}{4}$

Thus the general solution is

$$y = c_1e^t + c_2te^t + (-\frac{t^2 ln(t)}{2} + \frac{t^2}{4})e^t + (tln(t) - t)te^t$$

In general, to determine if there is a unique solution to the IVP, $y'' - 4y' + 4y = 0$, $y(x_0) = y_0$, $y'(x_0) = y_1$, we solve for unknowns a_0 and a_1.

$$y(x_0) = a_0\phi_0(x_0) + a_1\phi_1(x_0)$$
$$y'(x_0) = a_0\phi_0'(x_0) + a_1\phi_1'(x_0)$$

Note that the above system of two equations has a unique solution for the two unknowns a_0 and a_1 if and only if $det \begin{pmatrix} \phi_0(x_0) & \phi_1(x_0) \\ \phi_0'(x_0) & \phi_1'(x_0) \end{pmatrix} \neq 0$

In other words the IVP has a unique solution iff the Wronskian of ϕ_0 and ϕ_1 evaluated at x_0 is not zero. Recall that by theorem , this also implies that ϕ_0 and ϕ_1 are linearly independent and hence the general solution is $y = a_0\phi_0(x) + a_1\phi_1(x)$ by theorem.

Show that $\phi_0(x) = (-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}((n-1))}{n!}x^n$ and $\phi_1(x) = \Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n$ are linearly independent by calculating the Wronskian of these two functions evaluated at $x_0 = 0$.

$$W(\phi_1, \phi_2)(x) = \begin{pmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{pmatrix} = \begin{pmatrix} (-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n-1)}{n!}x^n & \Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n \\ (-2)\Sigma_{n=1}^{\infty}\frac{2^{n-1}(n-1)}{(n-1)!x^{n-1}} & \Sigma_{n=1}^{\infty}\frac{n2^{n-1}}{(n-1)!}x^{n-1} \end{pmatrix}$$

$$W(\phi_1, \phi_2)(0) = \begin{pmatrix} (-2)2^{0-1}(-1) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0$$

Hence $\phi_0(x) = (-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}((n-1))}{n!}x^n$ and $\phi_1(x) = \Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n$ are linearly independent

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When possible identify the functions giving the series solutions. Recall that by Taylor's theorem and the ratio test, $e^{2x} = \Sigma_{n=0}^{\infty}\frac{f^{(n)}(x)}{n!}x^n = \Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n$ for all x.

$$f(x) = a_1\Sigma_{n=0}^{\infty}\frac{n2^{n-1}}{n!}x^n - 2a_0\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n-1)}{n!}x^n$$

$$= a_1\Sigma_{n=0}^{\infty}\frac{n2^{n-1}}{n!}x^n - 2a_0\Sigma_{n=0}^{\infty}\frac{n2^{n-1}}{n!}x^n + 2a_0\Sigma_{n=0}^{\infty}\frac{2^{n-1}}{n!}x^n$$

$$= (a_1 - 2a_0)\Sigma_{n=0}^{\infty}\frac{n2^{n-1}}{n!}x^n + a_0\Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n$$

$$= (a_1 - 2a_0)x\Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^{n-1} + a_0\Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n$$

$$= (a_1 - 2a_0)x\Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n + a_0\Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n$$

$$= (a_1 - 2a_0)xe^{2x} + a_0e^{2x}$$

Note we have recovered the solution we found using the 3.4 method.

Note a power series solutions exists in a neighborhood of x_0 when the solution is analytic at x_0. I.e, the solution is of the form $y = \Sigma_{n=0}^{\infty}a_n(x - x_0)^n$ where this series has a nonzero radius of convergence about x_0.

When do we know an analytic solution exists? I.e, when is this method guaranteed to work?

Special case: $P(x)y'' + Q(x)y' + R(x)y = 0$

Then $y''(x) = -\frac{Q}{P}y' - \frac{R}{P}y$ $y'' + \frac{Q}{P}y' + \frac{R}{P}y$

Definition: The point x_0 is an *ordinary point* of the ODE,
$$P(x)y'' + Q(x)y' + R(x)y = 0$$
if $\frac{Q}{P}$ and $\frac{R}{P}$ are analytic at x_0.

Theorem 5.3.1: If x_0 is an ordinary point of the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$, then the general solution to this ODE is
$$y = \Sigma_{n=1}^{\infty}a_n(x - x_0)^n = a_0\phi_0(x) + a_1\phi_1(x)$$
where ϕ_i are power series solutions that are analytic at x_0. The solutions ϕ_0, ϕ_1 form a fundamental set of solutions. The radius of convergence for each of these series solutions is at least as large as the minimum radii of convergence of the series for $\frac{Q}{P}$ and $\frac{R}{P}$.

Theorem: If P and Q are polynomial functions, then $y = Q(x)/P(x)$ is analytic at x_0 if and only if $P(x_0) \neq 0$. Moreover if Q/P is reduced, the radius of convergence of $Q(x)/P(x) = min\{||x_0 - x|| \mid x \in \mathbb{C}, P(x) = 0\}$ where $||x_0 - x||$ = distance from x_0 to x in the complex plane.

If x_0 is an ordinary pt.
Translate eqn ~~so~~ so x_0 is at the origin
Then $x_0 = 0$ is an ordinary
Guess $y = \sum_{n=0}^{\infty} a_n x^n$

If x_0 is not an ordinary pt $\Rightarrow$ singular pt

Series Solutions Near a Regular Singular Point

MATH 365 *Ordinary Differential Equations*

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Spring 2015

http://banach.millersville.edu/~bob/math365/Singular/main.pdf

Background

We will find a power series solution to the equation:

$$P(t)y'' + Q(t)y' + R(t)y = 0.$$

We will assume that t_0 is a **regular singular point**. This implies:

1. $P(t_0) = 0$,
2. $\lim_{t \to t_0} \frac{(t - t_0)Q(t)}{P(t)}$ exists,
3. $\lim_{t \to t_0} \frac{(t - t_0)^2 R(t)}{P(t)}$ exists.

If regular singular point, can use ch 5 method

If not regular singular pt $\Rightarrow$ irregular $\Rightarrow$ out of luck (not covered in this class)

Simplification

If $t_0 \neq 0$ then we can make the change of variable $x = t - t_0$ and the ODE:

$$P(x + t_0)y'' + Q(x + t_0)y' + R(x + t_0)y = 0.$$

has a regular singular point at $x = 0$.

From now on we will work with the ODE

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

having $\boxed{\text{a regular singular point at } x = 0.}$

Assumptions (1 of 2)

Since the ODE has a regular singular point at $x = 0$ we can define

$$x\frac{Q(x)}{P(x)} = xp(x) \quad \text{and} \quad x^2\frac{R(x)}{P(x)} = x^2q(x)$$

which are analytic at $x = 0$ and

$$\lim_{x \to 0} \frac{xQ(x)}{P(x)} = \lim_{x \to 0} xp(x) = p_0$$

$$\lim_{x \to 0} \frac{x^2R(x)}{P(x)} = \lim_{x \to 0} x^2q(x) = q_0.$$

Practical (ie you may need to do this)

Assumptions (2 of 2)

Furthermore since $xp(x)$ and $x^2q(x)$ are analytic,

$$xp(x) = \sum_{n=0}^{\infty} p_nx^n$$

$$x^2q(x) = \sum_{n=0}^{\infty} q_nx^n$$

for all $-\rho < x < \rho$ with $\rho > 0$.

Motivation

Re-writing the ODE

The second order linear homogeneous ODE can be written as

$$\begin{aligned}
0 &= P(x)y'' + Q(x)y' + R(x)y \\
&= y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y \\
&= x^2y'' + x^2\frac{Q(x)}{P(x)}y' + x^2\frac{R(x)}{P(x)}y \\
&= x^2y'' + x[xp(x)]y' + [x^2q(x)]y \\
&= x^2y'' + x[p_0 + p_1x + \cdots + p_nx^n + \cdots]y' \\
&\quad + [q_0 + q_1x + \cdots + q_nx^n + \cdots]y.
\end{aligned}$$

$\frac{Q}{P} = p$

$\frac{R}{P} = q$

$y'' + py' + qy$

$x^2y'' + x[xp(x)]y' + [x^2q(x)]y$

Special Case: Euler's Equation

$0 = x^2 y'' + x\left[\sum_0^\infty p_n x^n\right] y' + \left[\sum_0^\infty q_n x^n\right] y$

If $p_n = 0$ and $q_n = 0$ for $n \geq 1$ then

$$\begin{aligned} 0 &= x^2y'' + x\,[p_0 + \cancel{p_1 x + \cdots + p_n x^n + \cdots}]\,y' \\ &\quad + [q_0 + \cancel{q_1 x + \cdots + q_n x^n + \cdots}]\,y \\ &= x^2 y'' + p_0 x y' + q_0 y \end{aligned}$$

which is Euler's equation.

$0 = x^2 y'' + p_0 x y' + q_0 y$

Guess: $y = x^r$

$x^2 (r)(r-1) x^{r-2} + p_0 x r x^{r-1} + q_0 x^r = 0$

$[r(r-1) + p_0 r + q_0] x^r = 0$ only one term x^r

$\Rightarrow r(r-1) + p_0 r + q_0 = 0$ solve for r etc

General Case

When $p_n \neq 0$ and/or $q_n \neq 0$ for some $n > 0$ then we will assume the solution to

$$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0$$

has the form

Guess

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n},$$

an Euler solution multiplied by a power series.

If not Euler for regular singular value

Solution Procedure

Assuming $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$ we must determine:

1. the values of r, ← new for regular singular value
2. a recurrence relation for a_n,
3. the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$

similar to ordinary but w/ r

Example (1 of 8)

Consider the following ODE for which $x = 0$ is a regular singular point.

$$4xy'' + 2y' + y = 0$$

Assuming $y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$ is a solution, determine the values of r and a_n for $n \geq 0$.

Example (1 of 8)

Consider the following ODE for which $x = 0$ is a regular singular point.

$$4xy'' + 2y' + y = 0$$

Assuming $y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$ is a solution, determine the values of r and a_n for $n \geq 0$.

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} \\ y''(x) &= \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2} \end{aligned}$$

Example (2 of 8)

$$\begin{aligned} 0 &= 4xy'' + 2y' + y \\ &= 4x \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2} + 2 \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} \\ &\quad + \sum_{n=0}^{\infty} a_n x^{r+n} \\ &= \sum_{n=0}^{\infty} 4(r+n)(r+n-1) a_n x^{r+n-1} + \sum_{n=0}^{\infty} 2(r+n) a_n x^{r+n-1} \\ &\quad + \sum_{n=0}^{\infty} a_n x^{r+n} \\ &= \sum_{n=0}^{\infty} [4(r+n)(r+n-1) + 2(r+n)]\, a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \end{aligned}$$

$\boxed{\text{5.4: Euler equation: } x^2y'' + \alpha xy' + \beta y = 0}$

Let $L(y) = x^2y'' + \alpha xy' + \beta y$

Recall that L is a linear function and if f is a solution to the euler equation, then $L(f) = 0$.

Note that if $x \neq 0$, then x is an ordinary point and if $x = 0$, then x is a singular point.

Suppose $x > 0$. Claim $L(x^r) = 0$ for some value of r

$y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$

$x^2y'' + \alpha xy' + \beta y = 0$

$x^2 r(r-1)x^{r-2} + \alpha x r x^{r-1} + \beta x^r = 0$

$(r^2 - r)x^r + \alpha r x^r + \beta x^r = 0$

$x^r[r^2 - r + \alpha r + \beta] = 0$

$x^r[r^2 + (\alpha - 1)r + \beta] = 0$

Thus x^r is a solution iff $r^2 + (\alpha - 1)r + \beta = 0$

Thus $r = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$

Suppose $x < 0$. Claim $L((-x)^r) = 0$ for some value of r

$y = (-x)^r$, $y' = -r(-x)^{r-1}$, $y'' = r(r-1)(-x)^{r-2}$

$x^2y'' + \alpha xy' + \beta y = 0$

$x^2 r(r-1)(-x)^{r-2} - \alpha x r(-x)^{r-1} + \beta(-x)^r = 0$

$(r^2 - r)(-x)^r + \alpha r(-x)^r + \beta(-x)^r = 0$

$(-x)^r[r^2 - r + \alpha r + \beta] = 0$

$(-x)^r[r^2 + (\alpha - 1)r + \beta] = 0$

Thus $(-x)^r$ is a solution iff $r^2 + (\alpha - 1)r + \beta = 0$

Thus $r = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$

Recall $|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$

Thus $|x|^r = \begin{cases} x^r & \text{if } x > 0 \\ (-x)^r & \text{if } x < 0 \end{cases}$

Thus if $r = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$, then $y = |x|^r$ is a solution to Euler's equation for $x \neq 0$.

for Euler eqn case

Case 1. 2 real distinct roots, r_1, r_2:
General solution is $y = c_1|x|^{r_1} + c_2|x|^{r_2}$.

Case 2: 2 complex solutions $r_i = \lambda \pm i\mu$:

Convert solution to form without complex numbers.

Note $|x|^{\lambda \pm i\mu} = e^{ln(|x|^{\lambda \pm i\mu})} = e^{(\lambda \pm i\mu) ln|x|} = e^{\lambda ln|x|} e^{i(\pm \mu ln|x|)}$

$= |x|^\lambda[cos(\pm \mu ln|x|) + isin(\pm \mu ln|x|)]$

$= |x|^\lambda[cos(\mu ln|x|) \pm isin(\mu ln|x|)]$

$|x|^\lambda \cdot |x|^{\pm i\mu}$

Case 3: 1 repeated root: Find 2nd solution. ?