Ch 7 and 9

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by

$$x_1'(t) = ax_1 + bx_2,$$
$$x_2'(t) = cx_1 + dx_2$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

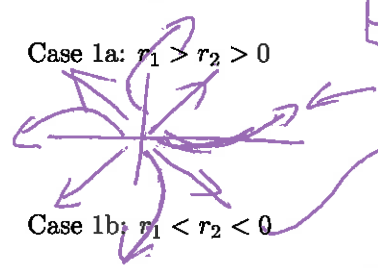
$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc = 0.$$

$$\text{Thus } r = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

Case 1: $(a+d)^2 - 4(ad-bc) > 0$ Two real solns

Hence the general solutions is $\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{r_2 t}}.$

Case 1a: $r_1 > r_2 > 0$

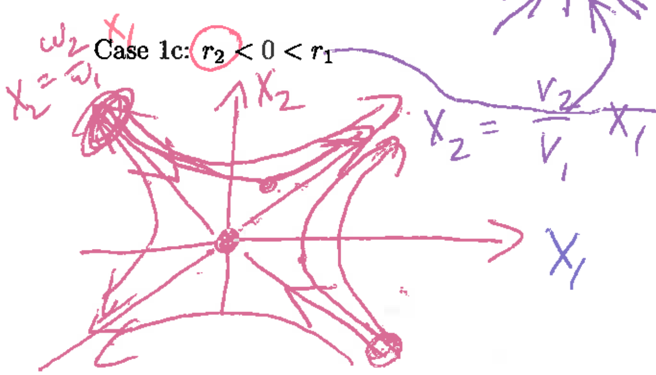


$\vec{X} = 0$ ~~is~~ equil soln is a nodal source

Case 1b: $r_1 < r_2 < 0$

$\vec{X} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ equil soln is a nodal sink

Case 1c: $r_2 < 0 < r_1$



$\vec{X} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the equilibrium saddle pt

Case 2: $(a+d)^2 - 4(ad-bc) = 0$ One repeated e. value

Case 2i: Two independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{rt}$

Case 2ii: One independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} t + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] e^{rt}$

Case 2a: $r > 0$

Case 2b: $r < 0$

Case 3: $(a+d)^2 - 4(ad-bc) < 0$. I.e., $r = \lambda \pm i\mu$

Suppose the eigenvector corresponding to this eigenvalue is

$$\begin{pmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Then general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 cos(\mu t) - w_1 sin(\mu t) \\ v_2 cos(\mu t) - w_2 sin(\mu t) \end{pmatrix} e^{\lambda t} + c_2 \begin{pmatrix} v_1 sin(\mu t) + w_1 cos(\mu t) \\ v_2 sin(\mu t) + w_2 cos(\mu t) \end{pmatrix} e^{\lambda t}$$

$$\boxed{c_1 \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cos \mu t - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \sin \mu t \right) e^{\lambda t} + c_2 \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \sin \mu t + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \cos \mu t \right) e^{\lambda t}}$$

Case 3a: $\lambda > 0$

Case 3a: $\lambda < 0$

Case 3a: $\lambda = 0$

$x' = A\vec{x}$ ← special case

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix} = (a-\lambda)^2 + b^2$$
$$= \lambda^2 - 2a\lambda + a^2 + b^2$$

Thus $\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2} = \frac{2a \pm \sqrt{-4b^2}}{2} = a \pm bi$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ implies $\begin{array}{l} x_1' = ax_1 + bx_2 \\ x_2' = -bx_1 + ax_2 \end{array}$

Change to polar coordinates: $r^2 = x_1^2 + x_2^2$ and $tan\theta = \frac{x_2}{x_1}$

Take derivative with respect to t of both equations:

$2rr' = 2x_1x_1' + 2x_2x_2'$ implies

$rr' = x_1(ax_1 + bx_2) + x_2(-bx_1 + ax_2)$

$= ax_1^2 + bx_1x_2 - bx_1x_2 + ax_2^2 = a(x_1^2 + x_2^2) = ar^2$

Thus $\cancel{r}r' = ar^{\cancel{2}}$ implies $\frac{dr}{dt} = ar$ and thus $\boxed{r = Ce^{at}}$.

$(sec^2\theta)\theta' = \frac{x_1x_2' - x_1'x_2}{x_1^2} = \frac{x_1(-bx_1+ax_2)-(ax_1+bx_2)x_2}{x_1^2}$

$= \frac{-bx_1^2 + ax_1x_2 - ax_1x_2 - bx_2^2}{x_1^2} = \frac{-b(x_1^2+x_2^2)}{x_1^2} = \frac{-b(r^2)}{x_1^2} = -bsec^2\theta$

$(\cancel{sec^2\theta})\theta' = -b\cancel{sec^2\theta}$ implies $\theta' = -b$ and thus $\boxed{\theta = -bt + \theta_0}$

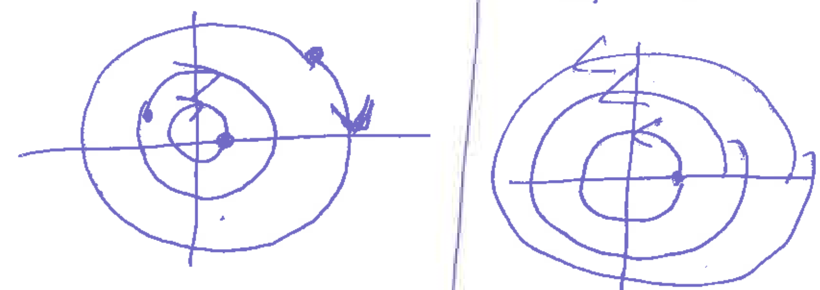
eigenvalue $\lambda = a \pm bi$

$a > 0 \Rightarrow r = Ce^{at}$

$b > 0$ spiral source $b < 0$

$a = 0 \Rightarrow r = Ce^{0t} = C$

$b > 0$ $b < 0$



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$a < 0$ $r = Ce^{at} \to 0$ as $t \to \infty$

$b > 0$ $b < 0$

spiral sink