Quiz 6 practice
April 27 2016

Show your work

[20] 1.) Given that $\mathcal{L}(e^{at}sin(bt)) = \boxed{\frac{b}{(s-a)^2+b^2}}$; find $\mathcal{L}^{-1}\left(\frac{4}{s^2+3s+10}\right) = \mathcal{L}^{-1}\left(\frac{}{s^2+3s+\frac{9}{4}-\frac{9}{4}+10}\right)$

$(s+\frac{3}{2})^2$

$$\frac{4}{\underbrace{(s+\frac{3}{2})^2-\frac{9}{4}+10}_{s^2+3s+\cancel{\frac{9}{4}}-\cancel{\frac{9}{4}}+10}} = \frac{\left(\frac{2}{\sqrt{31}}\right)4\left(\frac{\sqrt{31}}{2}\right)}{(s-(-\frac{3}{2}))^2+\frac{31}{4}}$$

Messy

$\Rightarrow a = -\frac{3}{2}$ $\quad b = \sqrt{\frac{31}{4}} = \frac{\sqrt{31}}{2}$

See Scratch paper

$\mathcal{L}^{-1}\left(\frac{4}{s^2+5s+100}\right) = \frac{8}{\sqrt{31}} e^{-\frac{3}{2}t} \sin\left(\frac{\sqrt{31}}{2}t\right)$

2.) Solve $\mathbf{x}' = \begin{pmatrix} 4 & 0 \\ 2 & 3 \end{pmatrix}\mathbf{x}$ e. values SINCE TRIANGULAR MATR

Find e.values: $|A-\lambda I| = \begin{vmatrix} 4-\lambda & 0 \\ 2 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda)-0=0$

$\Rightarrow \lambda = 3, 4$

$\lambda=3$: $A-3I = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow$ e. vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda=4$: $A-4I \Rightarrow \begin{pmatrix} 0 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ e. vector $= \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\mathbf{x} = c_1\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{4t}$

$x_1' = 4x_1 + x_2$

$x_2' = 5x_1 + 0$

Solve: $\vec{x}\,' = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}$ has e.vectors $c_1 \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ w/ e.value $= -1$

and e.vectors $c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ w/ e. value $= 5$

Thus general solution is

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}}$$

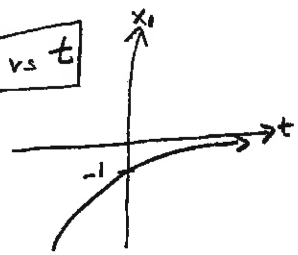
I.V.P.: Suppose $\vec{x}(0) = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

$\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \vec{x}(0) = c_1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^0$

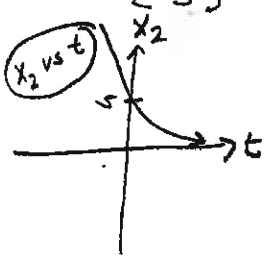
$$\left.\begin{aligned} -1 &= -c_1 + c_2 \\ 5 &= 5c_1 + c_2 \end{aligned}\right\} \Rightarrow c_1 = 1,\ c_2 = 0$$

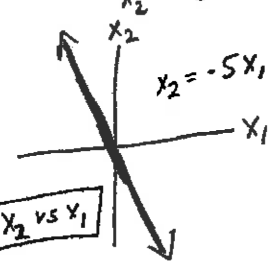
If $\vec{x}(0) = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} e^{-t} \Rightarrow \begin{aligned} x_1 &= -e^{-t} \\ x_2 &= 5e^{-t} \end{aligned}$

$\boxed{x_1 \text{ vs } t}$




$x_2$ vs $t$



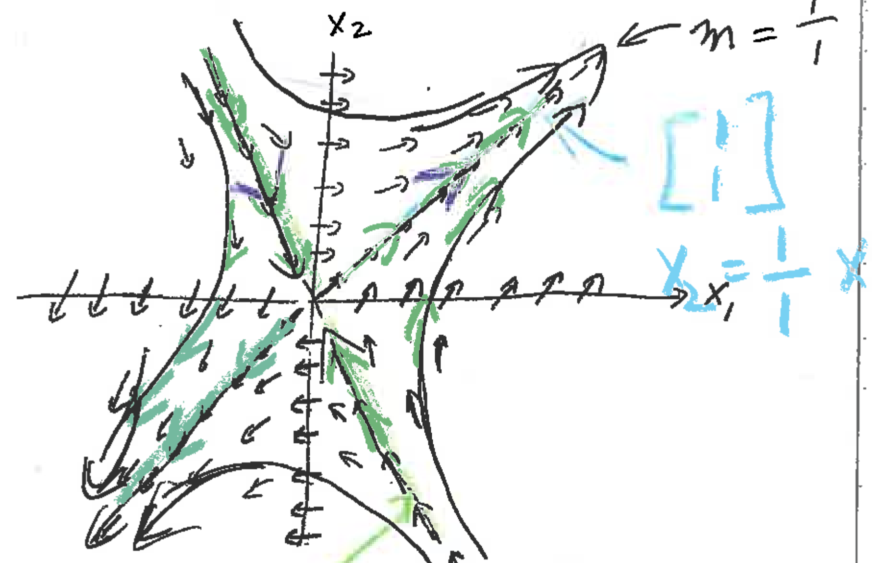

$\boxed{x_2 \text{ vs } x_1}$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4x_1 + x_2 \\ 5x_1 \end{bmatrix}$$

Slopes

$$\boxed{\frac{dx_2}{dx_1} = \frac{dx_2}{dt} \cdot \frac{dt}{dx_1} = \frac{dx_2/dt}{dx_1/dt} = \frac{x_2'}{x_1'}}$$

$$= \boxed{\frac{5x_1}{4x_1 + x_2}}$$




If $x_2 = -5x_1 \Rightarrow \frac{x_2'}{x_1'} = \frac{5x_1}{4x_1 - 5x_1} = \frac{5x_1}{-x_1} = -5$

$c_2 = 0 \Rightarrow \begin{bmatrix} -1 \\ 5 \end{bmatrix} \Rightarrow x_2 = \frac{-5}{1} x_1$

$x_1 = -ce^{-t}$

$x_2 = c_1 5e^{-t} = 5(-x_1)$

Ch 7 and 9

Suppose an object moves in the 2D plane (the $x_1, x_2$ plane) so that it is at the point $(x_1(t), x_2(t))$ at time $t$. Suppose the object's velocity is given by

$$x_1'(t) = ax_1 + bx_2,$$
$$x_2'(t) = cx_1 + dx_2$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc = 0.$$

Thus $r = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

$\boxed{\textbf{Case 1: } (a+d)^2 - 4(ad-bc) > 0}$ 2 real e. values

Hence the general solutions is $\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{r_2 t}}$

Case 1a: $r_1 > r_2 > 0$

Case 1b: $r_1 < r_2 < 0$

Case 1c: $r_2 < 0 < r_1$

**Case 2:** $(a+d)^2 - 4(ad-bc) = 0$ ← 1 repeated e. value

Case 2i: Two independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{rt}$

Case 2ii: One independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \left[ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} t + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] e^{rt}$

Case 2a: $r > 0$

Case 2b: $r < 0$

**Case 3:** $(a+d)^2 - 4(ad-bc) < 0$. I.e., $r = \lambda \pm i\mu$ ← 2 complex e. values

Suppose the eigenvector corresponding to this eigenvalue is

$$\begin{pmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Then general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 cos(\mu t) - w_1 sin(\mu t) \\ v_2 cos(\mu t) - w_2 sin(\mu t) \end{pmatrix} e^{\lambda t} + c_2 \begin{pmatrix} v_1 sin(\mu t) + w_1 cos(\mu t) \\ v_2 sin(\mu t) + w_2 cos(\mu t) \end{pmatrix} e^{\lambda t}$$ [redacted]

Case 3a: $\lambda > 0$

Case 3a: $\lambda < 0$

Case 3a: $\lambda = 0$

$$\vec{X} = e^{\lambda t} \left[ c_1 \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cos\mu t - \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \sin\mu t \right\} + c_2 \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \sin\mu t + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \cos\mu t \right\} \right]$$